Core of Convex Matching Games: A Scarf's Lemma Approach*

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Abstract

The core of a matching game is often empty when the market does not have a two-sided structure, contracts are multilateral, or agents have complementary preferences. In this paper, I use Scarf's lemma to show that given a convexity structure that I introduce, the core of a matching game is always nonempty, even if the game has an arbitrary contracting network, multilateral contracts, and complementary preferences. I provide three applications to show how the convexity structure is satisfied in different contexts by different assumptions. In the first application, I show that in large economies, the convexity structure is satisfied if the set of participants in each contract is small compared to the overall economy. Remarkably, no restriction on agents' preferences is needed beyond continuity. The second application considers finite economies, and I show that the convexity structure is satisfied if all agents have convex, but not necessarily substitutable, preferences. The third application considers a large-firm, many-to-one matching market with peer preferences, and I show that the convexity structure is satisfied under convexity of preferences and a competition aversion restriction on workers' preferences over colleagues. Because of the convexity structure, all three applications have a nonempty core.

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1 Introduction

Although matching theory has been successfully applied to markets with indivisible goods, personalized contract terms, and non-transferable or imperfectly transferable payoffs, the literature has primarily focused on two-sided markets with bilateral contracts and substitutable preferences[1]. These restrictions significantly limit the scope of existing matching models because they are often not satisfied in reality. For example, in a labor market with firms and workers, substitutable preferences are not satisfied if firms demand workers with complementary skills or dual-career couples demand two jobs in the same region. Moreover, the two-sided structure of the market is violated if firms can create joint ventures or workers can build economic or social relationships, such as marriage or labor unions. In these cases, relationships occur between firms and workers, as well as among firms and among workers. Furthermore, multilateral relationships or contracts naturally emerge when we consider joint ventures or projects that demand a rich set of resources or skills and thus involve more than two parties. Going one step further, as workers usually value not only the firm for which they work but also the colleagues with whom they work, we may interpret the nature of a firm as a multilateral economic relationship among all workers in the firm. Under this interpretation, a probably more appropriate approach is to model the labor market as a coalition formation game[2] instead of a two-sided market. In a coalition formation game, firms are no longer exogenous institutions but can be endogenously restructured, liquidated, or created by their workers. Furthermore, because individuals in reality may be simultaneously involved in multiple economic and social relationships, we also need to go beyond the narrowly defined coalition formation games in the literature, in which each agent can join at most one coalition. Clearly, a labor market with all the complications discussed above is far beyond the scope of standard matching models that are restricted to two-sided markets with bilateral contracts and substitutable preferences.

The major difficulty that arises when relaxing these restrictions is the empty core problem. With arbitrary contracting networks, multilateral contracts, or complementary preferences, it is well known that the core of a matching model is often empty—that is, there is no allocation immune to profitable joint deviations by groups of agents. For example, Gale and Shapley (1962) highlight the possibility of an empty core in a one-sided market using their unstable roommate example. In Kelso and Crawford (1982), substitutable preferences are shown to be indispensable for the nonemptiness of the core in a two-sided, many-to-one matching model. Moreover, Alkan (1988) shows that the core may be empty with multilateral

[1]See, for example, Kelso and Crawford (1982), Roth (1984), Hatfield and Milgrom (2005).

[2]See, for example, Banerjee, Konishi, and Tayfun (2001), Cechlarova and Romero-Medina (2001), and Bogomolnaia and Jackson (2002).

contracts, even if all agents have additively separable preferences. When the core is empty, every allocation is considered unstable in the sense that there is always a group of agents who can benefit from ignoring the prescribed allocation and instead taking some joint action by themselves. Therefore, the possibility of having an empty core is a fundamental problem when we apply a matching model to either a decentralized or centralized market. In a decentralized market, the empty core problem renders our model useless because it has no predictive power. On the other hand, when we explore the possibility of centralizing a market from a mechanism design perspective, the empty core problem is a substantial threat to whatever mechanism we might devise, as there is always a group of agents who can benefit from bypassing the mechanism and instead acting according to some agreement reached by themselves. Therefore, if we can resolve the empty core problem faced when considering arbitrary contracting networks, multilateral contracts, and complementary preferences, the scope and applicability of matching models will be greatly expanded to markets with more complicated economic and social relationships.

The goal of this paper is motivated by the discussions above. Specifically, I explore the possibility of obtaining nonempty core results in models that allow for general contracting network structures, multilateral contracts, and complementary preferences. Moreover, I also want to maintain, if possible, the strengths of existing matching models, including the flexibility to cope with indivisible goods, personalized contract terms, and non-transferable or imperfectly transferable payoffs.

In this paper, I show that the core is nonempty in all matching games with a convexity condition that I will introduce, including a large class of models that allow for arbitrary contracting networks, multilateral contracts, and complementary preferences. In a rough sense, the convexity of a matching game requires that the allocation space is convex and that for each potential block, the set of unblocked allocations is also convex. As we will see, this convexity of matching games is not directly related to convex preferences. In some applications, the matching game is convex without an assumption of convex preferences, while in other applications, convex preferences are not sufficient to guarantee the convexity of the matching game. Furthermore, my notion of convex matching games is not related to the notion of convex cooperative games in Shapley (1971), which requires the characteristic function to be supermodular. The notion of convex matching games that I will introduce can handle models in which the characteristic function fails to exhibit supermodularity.

I provide three applications of the nonempty core result; convexity is satisfied by a different set of assumptions in each application. In the first application, I consider a continuum large-economy model with small contracts, in the sense that the set of participants in each contract is small compared to the economy as a whole. Each contract is interpreted as an

economic or social relationship among a group of agents, such as employment, school enrollment, joint venture, or marriage. Given finitely many types of agents and continuous preferences, I show that the core is always nonempty even with arbitrary contracting networks, multilateral contracts, and arbitrary, possibly complementary, preferences. In this application, the convexity condition is satisfied because of the assumptions of small contracts, and the convexity of preferences is not relevant. This model is closely related to one of the results[3] in Azevedo and Hatfield (2015), but my model is more general in that the set of contract terms is allowed to be rich and agents are allowed to have continuous preferences over a continuum of alternatives. The model in Azevedo and Hatfield (2015), by contrast, is restricted to discrete contract terms, and agents are assumed to have strict preferences over only finitely many alternatives. A rich set of contract terms in my model offers greater flexibility, in particular, to subsume transferable utility (TU) models by letting each contract contain a term that specifies monetary transfers for all participants and letting all agents' utility functions be quasi-linear in money. Furthermore, each contract in Azevedo and Hatfield (2015) is assumed to have only finitely many participants. Therefore, their large-economy model can only be viewed as a limit of a sequence of economies where the number of agents per contract is fixed, while the number of contracts grow proportionally to the size of the economy. In my model, however, each contract is allowed to involve a continuum of agents as long as the continuum has zero mass. This may provide a new way to approximate a large economy, in which both the number of agents per contract and the number of contracts grow sub-linearly with respect to the size of the economy.

In the second application, I show that the core is nonempty in a finite economy with multilateral contracts if all agents have convex and continuous preferences. The convexity of this matching game is satisfied in a straightforward way because of the convexity of preferences. In this model, the terms of each contract can vary continuously within a convex set, and each agent is assumed to have convex, but not necessarily substitutable, preferences over terms of contracts that involve him or her. In the contexts of consumption and production, having convex preferences over contract terms corresponds to quasi-concave utility functions of consumption and quasi-convex cost functions of production. When applied to pure exchange economies, the result reduces to the classic one that the core is nonempty given convex preferences, but my model is more general because goods are allowed to be

[3]Remarkably, Azevedo and Hatfield (2015) offer three distinct results. Their first result (Section 4) is the existence of stable matching in a continuum, two-sided, many-to-many matching market, provided that agents on one side have substitutable preferences. Their second result (Section 5) is a nonempty core in a continuum economy with a general contracting network, multilateral contracts, and arbitrary preferences, which is closely related to, but less general than, my first application. Their third result (Section 6) is the existence of competitive equilibria in a continuum economy with transferable utility.

agent-specific, which is a common feature shared by existing matching models with contracts (Hatfield and Milgrom 2005). However, the assumption of a convex set of contract terms in my model usually implies continuously divisible goods, in contrast to standard matching models, which are able to handle indivisible goods. My model is related to the multilateral-contract model in Hatfield and Kominers (2015), where competitive equilibrium and, therefore, core allocations are shown to exist under TU and concave utility functions. By contrast, my model does not assume TU and can therefore be applied to markets in which utilities are imperfectly transferable. Another subtle difference is that convexity of preferences in my model corresponds to quasi-concavity of utility functions, which is weaker than concavity assumed in Hatfield and Kominers (2015).

In the third application, I study a large-firm, many-to-one matching model with peer preferences. There are finitely many firms and a continuum of workers in the market, and each firm is large in the sense that it can hire a continuum of workers. Workers may have preferences over their peers, in the sense that they value not only the firm for which they work but also the colleagues with whom they work. With peer preferences, each contract in this model is multilateral because it involves a firm and the set of all workers that firm employs. Because firms may hire a continuum of workers with positive mass, the assumption of small contracts in the first application is not satisfied, and therefore, this model is not a special case of the first application. In this model, I show that the core is nonempty if all firms and workers have convex and continuous preferences, and in addition, all workers' preferences over peers satisfy a "competition-aversion" condition. Roughly speaking, the competition-aversion condition requires that each worker does not like colleagues of his or her own type, possibly because workers of the same type have to compete for projects, resources, and promotions when employed by the same firm. A striking observation is that the core may be empty without the competition aversion condition, even if all firms and workers have convex and continuous preferences. As we will see, the convexity of this matching game is only satisfied when the convexity of preferences is combined with competition aversion. This model is related to that of Che, Kim, and Kojima (2017), which shows that stable matchings always exist in a large-firm, many-to-one matching model without peer preferences, provided that all firms have continuous and convex, but not necessarily substitutable, preferences. By contrast, my model additionally allows for peer preferences and highlights the importance of competition aversion for a nonempty core, although without peer preferences, my notion of the core is slightly different from the notion of stable matchings in Che, Kim, and Kojima (2017). Furthermore, their paper allows for a compact space of worker types, but I only consider finitely many types of workers.

Scarf's lemma is central to my approach to the nonemptiness of the core, in contrast

to the standard fixed-point approach in matching theory (see, for example, Adachi 2000, Fleiner 2003, Echenique and Oviedo 2004, 2006, and Hatfield and Milgrom 2005). Scarf's lemma first appeared in the seminal paper Scarf (1967), where it is used to show that a balanced non-transferable utility (NTU) game always has a nonempty core. The lemma has received attention from the combinatorics literature since Aharoni and Holzman (1998). In particular, Aharoni and Fleiner (2003) use Scarf's lemma to prove a fractional version of stable matchings for hypergraphs, which suggests a potential extension of Gale and Shapley (1962) to general contracting networks if fractional matchings in their paper can be interpreted appropriately as actual matchings. Relating to their paper, the convex matching games I will introduce can be viewed as a systematic framework for interpreting fractional allocations, and the applications in this paper provide concrete ways in which the interpretation may work in different contexts. My first and third applications provide a natural interpretation of fractional allocations in continuum economies, and my second application provides an interpretation when contract terms are taken from a convex set. Nguyen and Vohra (2017) study a finite two-sided labor market with couples and use Scarf's lemma to find a stable fractional matching that may not be interpreted as an actual matching. The next step in their paper is to use a rounding algorithm to find a nearby integer matching that is near-feasible and stable. By contrast, in the convex matching game defined in my paper, the stable fractional allocation found by Scarf's lemma can be directly interpreted as an actual allocation that is in the core, and therefore, a second step such as that in Nguyen and Vohra (2017) is unnecessary.

The remainder of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the framework of convex matching games and states the nonempty core result. Before proving this result, I examine three applications. Section 4 studies a general large-economy model with small contracts. Section 5 considers a finite-economy model with convex preferences. Section 6 studies a large-firm many-to-one matching model with peer preferences. All three applications are shown to be a convex matching game, as defined in Section 3, and therefore have a nonempty core as a corollary. Section 7 uses Scarf's lemma to prove the nonemptiness of the core in convex matching games, and Section 8 concludes.

2 Literature

In the matching literature, a number of papers have attempted to obtain existence results for core-like solution concepts while relaxing some of the standard restrictions, but they lack the level of generality offered in this paper. For example, some papers have been devoted to labor market matchings with complementary preferences, especially those that

emerge because of the presence of double-career couples. These works show that stable matchings exist only under very restrictive assumptions on preferences in finite markets (see, for example, Cantala 2004 and Klaus and Klijn 2005). In large markets, however, the problem of complementary preferences tends to vanish, as noted in a sequence of recent papers, including Kojima, Pathak, and Roth (2013), Ashlagi, Braverman, and Hassidim (2014), Azevedo and Hatfield (2015), and Che, Kim, and Kojima (2017). However, all of these results, except for Azevedo and Hatfield (2015), only apply to two-sided markets with bilateral contracts, which is a restriction I wish to relax in this paper.

Some papers attempt to go beyond two-sided markets, and they typically find that some restrictions on preferences or on the network structure have to be imposed to allow the existence of the core or stable allocations. For example, in an NTU finite market framework, Ostrovsky (2008) and Hatfield and Kominers (2012) show that stable matchings exist in a vertical supply chain network if preferences are fully substitutable. In a TU finite-market framework, Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013) show that competitive equilibria and core allocations exist in an arbitrary trading network given substitutable preferences. In a TU continuum-economy framework, Azevedo, Weyl, and White (2013) and Azevedo and Hatfield (2015) (in their Section 6) show that when substitutable preferences are relaxed, competitive equilibria and core allocations still exist. However, in my three applications in this paper, I assume neither TU, nor substitutability of preferences, nor a vertical supply chain network. Furthermore, all results mentioned in this paragraph only apply to bilateral contracts, but all three applications in my paper involve multilateral contracts.

Some papers in the literature consider multilateral contracts and find that a nonempty core can only be obtained under relatively restrictive assumptions on preferences or on the contracting network. For example, Dutta and Masso (1997) and Bodine-Baron, Lee, Chong, Hassibi, and Wierman (2011) study multilateral contracts that emerge as a result of preferences over peers in many-to-one matching problems. More generally, Banerjee, Konishi, and Tayfun (2001), Cechlarova and Romero-Medina (2001), Bogomolnaia and Jackson (2002), Papai (2004), and Pycia (2012) consider coalition formation games, in which agents endogenously form disjoint coalitions. In this framework, each coalition can be viewed as a multilateral contract, but each agent only demands at most one contract. The first and the second applications in my paper are more general in the sense that each agent may be simultaneously involved in multiple contracts. Furthermore, in my first application, no assumption on preferences beyond continuity is needed for the nonempty core result, as opposed to the restrictive assumptions made in the literature.

As discussed in the Introduction, the three applications in this paper are most related

to Section 5 of Azevedo and Hatfield (2015), Hatfield and Kominers (2015), and Che, Kim, and Kojima (2017), respectively. Further details on their relationship to these papers will be provided when I formally discuss each application. Notably, my first application is also related to a sequence of papers by Myrna Wooders. In Wooders (1983), Shubik and Wooders (1983), and Kovalenkov and Wooders (2003), an approximate notion of the core is shown to be nonempty in a large, finite cooperative game under mild regularity assumptions. In particular, the "small group effectiveness" condition in Kovalenkov and Wooders (2003) is similar to my small contract assumption. The small group effectiveness condition was first formulated in Wooders (1992) for TU cooperative games and later generalized to NTU games. It requires that "almost all gains to group formation can be realized by partitions of the players into groups bounded in absolute size", and it is sufficient for a nonempty approximate core in large, finite cooperative games. By contrast, the small contract assumption I introduce is for continuum economies and sufficient for the nonemptiness of the exact core. Remarkably, Kaneko and Wooders (1986) and Kaneko and Wooders (1996) prove the nonemptiness of the core for continuum NTU cooperative games. In these two papers, however, each allocation is assumed to be a partition of players into coalitions containing only finitely many players, and players across different coalitions are assumed to have no interaction. In my model, by contrast, a set of agents with positive mass may be connected, either directly or indirectly, through contracts, although each contract only involves a set of agents with zero mass. This is a more realistic assumption than partitioning a continuum economy into coalitions with only finitely many agents. Another difference is that in all papers by Wooders and her coauthors mentioned above, the model follows the tradition of cooperative game theory and starts from the set of feasible payoff vectors for each coalition as primitives. This approach abstracts from details on how coalitions of players function internally, while my approach follows the tradition of matching theory and considers these details by taking contracts as primitives.

3 Framework: Convex Matching Games

In this section, I introduce the concept of "convex matching games"and state the central result of this paper, i.e., a regular convex matching game always has a nonempty core. The proof of this result will be postponed to Section 7, after we study some applications. As we will see in the next three sections, the framework of convex matching games is applicable to a large class of models that allow for arbitrary contracting networks, multilateral contracts, and complementary preferences. Because of its generality, this framework has to be introduced with some level of abstractness. The exact meaning of each component of this framework

will depend on its context. In later sections, when we come to applications, the meaning of those abstract objects will become clear.

Consider a *matching game* $G := \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$. The set I is a finite set of *players*. In applications, a player $i \in I$ may represent either one agent or a continuum of identical agents, i.e., a type of agents. The set $\mathcal{M}$ is the set of *allocations*. For each allocation $\mu \in \mathcal{M}$, the vector $\phi(\mu) \in [0,1]^I$ is the *characteristic vector* of allocation μ. In applications in which $i \in I$ represents one agent, the characteristic value $\phi_i(\mu)$ is either 0 or 1, indicating whether agent i is "involved" in allocation μ. In applications in which $i \in I$ represents a continuum of identical players, the value $\phi_i(\mu) \in [0,1]$ represents the fraction of type-i agents involved in allocation μ. Let the set

$$\mathcal{M}_i := \{\mu \in \mathcal{M} : \phi_i(\mu) > 0\}$$

be the set of allocations that involve player i. Each player i is associated with a domination relation $\sqsupset_i$ from $\mathcal{M}_i$ to $\mathcal{M}$. When $\hat{\mu} \sqsupset_i \mu$, we say that $\hat{\mu} \in \mathcal{M}_i$ dominates allocation $\mu \in \mathcal{M}$ at player i. In the set $\mathcal{M}$ of allocations, let us assume that there exists a unique "empty" allocation μ^0 that involves no player, i.e., its characteristic vector $\phi(\mu^0) = \mathbf{0}$. In applications, μ^0 is the allocation under which agents do not interact with one another.

In applications, the domination relations $(\sqsupset_i)_{i \in I}$ are typically derived from agents' preferences. When i represents one agent, $\hat{\mu} \sqsupset_i \mu$ means that agent i strictly prefers $\hat{\mu}$ to μ. When i represents a type of agents, I am particularly interested in the domination relation $\sqsupset_i$ such that $\hat{\mu} \sqsupset_i \mu$ if and only if there are some type-i agents under allocation μ who are willing to switch to the worst position in allocation $\hat{\mu}$. If all players involved in allocation $\hat{\mu}$ are willing to switch to $\hat{\mu}$ from μ, then allocation μ is unstable in the sense that it is vulnerable to the profitable joint deviation $\hat{\mu}$. When this is the case, we say that the allocation μ is *blocked* by $\hat{\mu}$. This motivates the following definition.

Definition 3.1 *In a matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$, an allocation $\mu \in \mathcal{M}$ is* ***blocked*** *by a nonempty allocation $\hat{\mu}$, if $\hat{\mu} \sqsupset_i \mu$ for each $i \in I$ with $\phi_i(\hat{\mu}) > 0$. An allocation μ is in the* ***core*** *if it is not blocked by any nonempty allocation.*

The goal of this paper is to obtain nonemptiness of the core, and I find the following convexity structure to be particularly relevant to that end.

Definition 3.2 *A matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$ is* ***convex*** *if it satisfies the following three requirements:*

(1) The allocation space $\mathcal{M}$ is a subset of a vector space over the real field, with the empty allocation $\mu^0 \in \mathcal{M}$ being the zero vector.

(2) Whenever $\sum_{j=1}^{m} w^j \phi(\mu^j) \leq \mathbf{1}$*, we have* $\sum_{j=1}^{m} w^j \mu^j \in \mathcal{M}$*, where* $w^j > 0$ *and* $\mu^j \in \mathcal{M}$ *for all* $j = 1, 2, \ldots, m$*.*

(3) For each $i \in I$*, there exists a complete and transitive relation* $\trianglerighteq_i$ *over* $\mathcal{M}_i$ *s.t.* $\hat{\mu} \not\trianglelefteq_i \sum_{j=1}^{m} w^j \mu^j$ *if* $\sum_{j=1}^{m} w^j \phi_i(\mu^j) = 1$ *and* $\mu^j \trianglerighteq_i \hat{\mu}$ *for each* j *with* $\mu^j \in \mathcal{M}_i$*, where* $w^j > 0$ *and* $\mu^j \in \mathcal{M}$ *for all* $j = 1, 2, \ldots, m$ *s.t.* $\sum_{j=1}^{m} w^j \phi(\mu^j) \leq \mathbf{1}$*.*

To facilitate interpretation of the convexity structure defined above, call $\mu := \sum_{j=1}^{m} w^j \mu^j$ a ϕ_i-convex combination of $(\mu^j)_{j=1}^{m}$ if $\sum_{j=1}^{m} w^j \phi_i(\mu^j) = 1$ and $\sum_{j=1}^{m} w^j \phi(\mu^j) \leq \mathbf{1}$. Then, statement (2) in the definition essentially requires that for each i the allocation space $\mathcal{M}$ is closed under the operation of taking a ϕ_i-convex combination. Statement (3) requires that $\hat{\mu}$ does not dominate a ϕ_i-convex combination of a set of $\trianglerighteq_i$-better allocations at player i. In applications, the relation $\trianglerighteq_i$ can be roughly interpreted as the preference relation of player i. When i represents a continuum of identical players, the relation $\trianglerighteq_i$ is obtained by comparing the worst position for type-i agents under two allocations, as we will see in the large-economy application in the next section. Intuitively speaking, the convexity of a matching game requires that if player i weakly prefers a set of allocations to a potential block, then player i is unwilling to participate in the block under any ϕ_i-convex combination of these allocations.

In addition to the convexity structure, let us also endow matching games with the following topological structure.

Definition 3.3 *A matching game* $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$ *is* ***regular*** *if the following hold:*

(1) The allocation space $\mathcal{M}$ *is a compact topological space.*

(2) For each $i \in I$*, the set* $\{\mu \in \mathcal{M} : \hat{\mu} \not\trianglelefteq_i \mu\}$ *of allocations unblocked by* $\hat{\mu}$ *is closed for each* $\hat{\mu} \in \mathcal{M}_i$*.*

The regularity condition defined above is relatively mild. Intuitively,[4] statement (2) requires that a sequence of allocations that are not dominated by $\hat{\mu}$ at player i cannot converge to an allocation that is dominated. In applications, this is typically satisfied by assuming continuous preferences.

Now let us state the central theorem of this paper.

Theorem 3.4 *In a regular and convex matching game* G*, the core is always nonempty.*

The discussion for now has been entirely abstract, and we cannot learn much from this nonempty core result unless we endow the abstract framework with concrete meanings.

[4] This intuition is not entirely precise because closedness is different from sequential closedness in general topological spaces. In this paper, however, all three applications have a metrizable space of allocations, and thus, closeness and sequential closedness are equivalent.

Therefore, I will study three concrete applications in the next three sections and postpone the proof of the theorem to the end of this paper. I show that all three applications fit into the framework of convex matching games and therefore have nonempty core.

4 Large Economies with Small Contracts

In this section, I show that a large-economy model with a continuum of agents is a regular and convex matching game and, therefore, has a nonempty core. The model allows for general contracting networks and multilateral contracts, the terms of which are allowed to vary continuously. While agents can hold a bundle of contracts, I only impose a continuity assumption on an agent's preferences over bundles. In particular, I do not assume agents' preferences to be substitutable. Utility is non-transferable in this model in general, but the model also accommodates the TU framework when each contract contains a term that specifies monetary transfers for all participants and agents' preferences are quasi-linear in money. Moreover, this model accommodates coalition formation games in the sense of Banerjee, Konishi, and Tayfun (2001), Cechlarova and Romero-Medina (2001), and Bogomolnaia and Jackson (2002) by restricting each agent to demand at most one contract at a time.

The nonempty core result only relies on three relatively mild assumptions. First, I assume that there are finitely many types of agents in the economy. Agents of the same type have the same preferences and are considered identical by all other agents. Second, I assume that all contracts in the economy are small compared to the economy as a whole, in the sense that the set of participants in each contract has mass of zero, while the set of all agents in the economy has positive mass. This assumption rules out global public goods that affect a large set of agents, but it still allows us to consider local public goods that affect a continuum of agents with zero mass. Third, I assume that preferences are continuous.

There is a finite set I of agent types in this economy. For each agent type $i \in I$, there exists a mass $m_i > 0$ of type-i agents. Let R be the set of all possible *roles*, where each role $r \in R$ specifies an agent's tasks and compensations in an economic or social relationship with other agents. For example, in the roommate matching context, a typical role $r \in R$ can be "being a type-1 agent in a type-1-2 match responsible for cleaning the kitchen but only paying 40% of the rent". Let us assume the set R of roles to be a compact metric space, and roles that are close under the metric contain similar tasks and compensations.

In this economy, roles performed by different agents are coordinated by *contracts*, and each *contract type* $x \in X$ represents a consistent combination of roles in a certain economic or social relationship among a group of agents. In applications, a contract may represent an employment relationship between a firm and a worker, an enrollment relationship between

a school and a student, a joint venture by several firms, a marriage relationship between two individuals, or any other relationship among people. Formally, each contract type x is a Borel measure over the set R of roles, which represents the quantity of each role $r \in R$ involved in a type-x contract.[5] Endow X with the weak-* topology, and let us assume that X is a compact set that does not contain the zero measure over R.[6]

Each agent in this economy may wish to simultaneously participate in multiple relationships with different groups of agents, in which case an agent will hold a bundle of multiple roles. Let us assume that all agents demand at most $N \in \mathbb{N}$ roles at one time, and therefore, an acceptable *bundle* β of roles is a Borel measure over R that assigns measure 1 to at most N, possibly duplicate, roles in R. Let $\mathfrak{B}$ be the set of all such bundles, and endow $\mathfrak{B}$ with the weak-* topology. Note that both a bundle β and a contract type x are a measure over R, but they are conceptually unrelated because a bundle β measures the number of roles held by one agent, while a contract of type x measures the number of roles contained in a contract, most likely to be held by different agents. Each agent type i is associated with a complete and transitive preference relation $\succsim_i$ over $\mathfrak{B}$. Assume $\succsim_i$ to be continuous in the sense that all upper contour sets and lower contour sets are closed in $\mathfrak{B}$. A bundle $\beta \in \mathfrak{B}$ is *individually rational (IR)* for agent type i if $\beta \succsim_i \mathbf{0}$, where $\mathbf{0}$ is the zero measure over R that represents the empty bundle. Let $\mathfrak{B}_i$ be the set of nonempty IR bundles for type-i agents.[7]

An *allocation* μ specifies, for each i, the mass of type-i agents holding each nonempty IR bundle $\beta \in \mathfrak{B}_i$, subject to some feasibility requirement. Formally, an allocation is $\mu := (\mu_i)_{i \in I}$, where μ_i is a Borel measure over the set $\mathfrak{B}_i$ of nonempty IR bundles for type-i agents. First, feasibility requires the measure μ_i to respect the total mass constraint $\mu_i(\mathfrak{B}_i) \leq m_i$ for each agent type i. Second, all roles present under the allocation $\mu = (\mu_i)_{i \in I}$ need to be able to fit into a set of contracts, i.e., there exists a Borel measure μ_x over the set X of

[5] The measure x may carry different units in different contexts. When x represents a type of contracts that only involve finitely many agents, the measure x is integer-valued, measuring the head count of each role. When x represents a type of contracts that contain a continuum of agents, the measure x is real-valued, carrying some proper unit for the continuum.

[6] The weak-* topology on X is the weakest topology that makes the linear functional $L_f : X \to \mathbb{R}$ defined as

$$L_f(x) := \int_R f dx$$

continuous in x for all continuous functions $f : R \to \mathbb{R}$. Under the weak-* topology, the compactness assumption requires no more than boundedness and closedness, by the Banach-Alaoglu theorem. See the Appendix for details.

[7] I can show that the set $\mathfrak{B}_i$ is compact. Closedness is because of the continuity of $\succsim_i$, and removing the empty bundle $\mathbf{0}$ from $\mathfrak{B}_i$ is not a threat to closeness because a sequence of nonempty bundles never converges to the empty bundle $\mathbf{0}$. See the Appendix for details.

contract types s.t. the following accounting identity holds:

$$\sum_{i\in I}\int_{\beta\in\mathfrak{B}_i}\beta d\mu_i=\int_{x\in X}xd\mu_x$$

In the accounting identity above, both sides are measures over the set R of roles.[8] The left-hand side calculates the quantity of roles from the perspective of agents, while the right-hand side calculates the same quantity from the perspective of contracts.

Define addition and scalar multiplication on allocations component-wise, i.e.,

$$\mu+\mu':=(\mu_i+\mu_i')_{i\in I}$$
$$\lambda\mu:=(\lambda\mu_i)_{i\in I}$$

but with the caveat that the set of feasible allocations is not always closed under these two operations. Among all feasible allocations, notice that there is an empty allocation μ^0 under which all agents hold the empty bundle and no contract is present, i.e., μ_i^0 is the zero measure.

The assumption of small contracts is implicit in the feasibility structure above. Notice that if μ is a feasible allocation, then $\lambda\mu$ with $0<\lambda<1$ is still a feasible allocation, in which the quantity of contracts of each type is multiplied by λ. This implies that there must be a continuum of contracts of each type under an allocation, and therefore, each contract can only involve a set of agents with zero mass. To see this in another way, suppose, by contradiction, that there exists a type of large contracts, each of which involves positive mass of agents. Then, an allocation can only contain finitely many contracts of this type. When μ represents an allocation with one such contract, $\mu/2$ cannot be interpreted as a feasible allocation, which contradicts the feasibility structure of our setup.

Now, we are in a position to define the notions of blocking and the core. Because a block may involve a rich set of contracts arranged in an arbitrary way, it is convenient to use nonempty allocations to represent blocks. For a nonempty allocation $\hat{\mu}$ to block an

[8] To formally understand the accounting identity, on the left-hand side, each β is a measure over R, and μ_i is a measure over βs. Therefore the integral $\int_{\beta\in\mathfrak{B}_i}\beta d\mu_i$ is again a measure of R. More formally, the integral $\int_{\beta\in\mathfrak{B}_i}\beta d\mu_i$ is defined as the linear functional $L_{\mu_i}:C(R)\to\mathbb{R}$

$$L_{\mu_i}(f):=\int_{\beta\in\mathfrak{B}_i}\left(\int_{r\in R}fd\beta\right)d\mu_i$$

where $C(R)$ is the set of continuous functions on R. By the mass constraint $\mu_i(\mathfrak{B}_i)\leq m_i$, the linear functional L_{μ_i} is bounded, and therefore, it is isomorphic to a finite Borel measure over R due to the Riesz representation theorem. Analogously, the integral on the right-hand side is also interpreted as a measure over R.

allocation μ, we require that if bundle $\hat{\beta}$ for type-i agents is present under the block $\hat{\mu}$, we need to find a positive mass of type-i agents under allocation μ who are willing to switch to $\hat{\beta}$. These type-i agents may come from two sources: agents who hold the empty bundle under allocation μ and agents who hold some nonempty bundle under μ that is strictly less preferred to $\hat{\beta}$. Following these intuitive descriptions, we have the following definition.

Definition 4.1 *An allocation μ is* ***blocked*** *by a nonempty allocation $\hat{\mu}$, if for each i with $\hat{\mu}_i(\mathfrak{B}_i) > 0$, we have either $\mu_i(\mathfrak{B}_i) < m_i$ or $\mu_i\left(\left\{\beta \in \mathfrak{B}_i : \hat{\beta} \succ_i \beta\right\}\right) > 0$ for all $\hat{\beta} \in Supp(\hat{\mu}_i)$. An allocation is in the* ***core*** *if it is not blocked by any nonempty allocation $\hat{\mu}$.*[9]

The model formulated above is closely related to the second result (Section 5) of Azevedo and Hatfield (2015), where the set R of roles, the set X of contracts, and the set of $\mathfrak{B}_i$ of IR bundles for type-i agents are assumed to be finite, and agents have strict preferences. This level of discreteness makes their model difficult to compare with, for example, TU models, in which monetary transfers can vary continuously. In contrast, the compact set of roles in my model offers greater flexibility to accommodate TU models by letting each contract contain a term that specifies monetary transfers for all participants and all agents' utility functions be quasi-linear in money. More generally, my model can also handle situations in which utility is continuously, but not perfectly, transferable, in which case the Pareto frontier is smooth but not linear.

Furthermore, each contract in Azevedo and Hatfield (2015) is assumed to have only finitely many participants. As a consequence, their model can only be interpreted as a limit of a sequence of economies where the number of agents per contract is fixed, while the number of contracts grows linearly with respect to the size of the economy. In my model, however, each contract may also involve a continuum of agents as long as the continuum has zero mass. Therefore, my model may provide another way to approximate a large economy, in which both the number of agents per contract and the number of contracts grow sub-linearly with respect to the size of the economy.

Now, let us consider a simplistic example to illustrate the model and clarify the notations.

Example 1 *Consider a continuum roommate problem with three types of agents, $I = \{1, 2, 3\}$; each type of agents has mass 1. Each pair of agents may choose to become roommates, but all agents only accept a roommate of a different type than their own. Further, type-i agents*

[9] Conceptually, a block is identified by a set of contracts of certain types assigned to agents in a certain way, and the mass of this set of contracts is irrelevant. Therefore, rigorously speaking, a block is the support of a nonempty allocation $\hat{\mu}$. In the definition, I directly use nonempty allocations to represent blocks for the sake of convenience, but bear in mind that $\hat{\mu}$ and $\lambda\hat{\mu}$, where λ is a small positive number, represent the same block. This also explains why in the definition we only need to find a positive mass, instead of a sufficient mass, of agents in the allocation μ who are willing to participate in the block $\hat{\mu}$.

strictly prefer having a type-$i+1$ roommate to having a type-$i-1$ roommate and prefer having a type-$i-1$ roommate to living alone.[10] *In this example, let us assume that no monetary transfer is allowed.*

In the terminology of the model, there are 6 *roles* $R = \{1/12, 2/12, 1/13, 3/13, 2/23, 3/23\}$, *where* i/ij *represents the role "being a type-i agent in a roommateship between a type-i and a type-j agent". There are* 3 *contract types* $X = \{12, 23, 13\}$, *where* $ij := \delta_{i/ij} + \delta_{j/ij}$ *represents the type of roommateship that involves one type-i and one type-j agent.*[11] *Because each agent can only join at most one roommateship, the set of acceptable bundles is simply* $\mathfrak{B} := \{\delta_r : r \in R\}$, *and the set of IR bundles for type-i agents is* $\mathfrak{B}_i = \left\{\delta_{i/i,i+1}, \delta_{i/i,i-1}\right\}$. *Moreover, we define* $\succsim_i$ *on* $\mathfrak{B}_i$ *s.t.* $\delta_{i/i,i+1} \succ_i \delta_{i/i,i-1}$.

An allocation is $\mu = \left((\mu_i)_{i=1}^3, \mu_x\right)$, *where* $\mu_i\left(\left\{\delta_{i/ij}\right\}\right)$ *is the mass of type-i agents in a type-ij roommateship, and* $\mu_x(\{ij\})$ *is the mass of type-ij roommateship. The total mass constraint requires that* $\mu_i\left(\left\{\delta_{i/i,i+1}\right\}\right) + \mu_i\left(\left\{\delta_{i/i,i-1}\right\}\right) \leq 1$, *and the accounting identity requires that* $\mu_i\left(\left\{\delta_{i/ij}\right\}\right) = \mu_x(\{ij\})$ *for each* (i, j), *where the left-hand side calculates the mass of role* i/ij *from the perspective of agents, and the right-hand side calculates the same mass from the perspective of contracts. Notice that the accounting identity implicitly requires that the mass* $\mu_i\left(\left\{\delta_{i/ij}\right\}\right)$ *of type-i agents in a type-ij roommateship is equal to the mass* $\mu_j\left(\left\{\delta_{j/ij}\right\}\right)$ *of type-j agents in a type-ij roommateship, as they are both required to be equal to the mass* $\mu_x(\{ij\})$ *of type-ij roommateship.*

Let us further use the example above to illustrate the notions of blocking and the core. Let μ^{ij} be the allocation under which each type-i or type-j agent is in a type-ij roommateship, and all agents of the third type are unmatched. Formally, we let $\mu_i^{ij}\left(\left\{\delta_{i/ij}\right\}\right) = \mu_j^{ij}\left(\left\{\delta_{j/ij}\right\}\right) = \mu_x^{ij}(\{ij\}) = 1$. Using Definition 4.1, it is straightforward to verify that the allocation $\mu^{i,i-1}$ is blocked by $\mu^{i,i+1}$. To see this, under the block $\mu^{i,i+1}$, only type i and type $i+1$ may hold a nonempty bundle, the only bundle for type i is $\delta_{i/i,i+1}$, and the only bundle for type $i+1$ is $\delta_{i+1/i,i+1}$. Under the allocation $\mu^{i,i-1}$, we can find a positive mass of type-i agents who are willing to accept the bundle $\delta_{i/i,i+1}$ since, in fact, all type-i agents are holding the less-preferred bundle $\delta_{i/i,i-1}$. We can also find a positive mass of type-$i+1$ agents who are willing to accept the bundle $\delta_{i+1/i,i+1}$ because, in fact, all type-$i+1$ agents are unmatched under the allocation $\mu^{i,i-1}$. Therefore, by Definition 4.1 the allocation $\mu^{i,i-1}$ is blocked by $\mu^{i,i+1}$, and thus, the allocation $\mu^{i,i-1}$ is not in the core. Therefore, none of the allocations μ^{12}, μ^{23}, or μ^{31} is in the core.

[10] In this example, let $3 + 1 := 1$ for indices of agent types.

[11] The measure δ_r over R is the Dirac measure that assigns measure 1 to r and measure 0 elsewhere.

There is a unique allocation μ^* in the core, which is defined as

$$\mu^* := \left(\mu^{12} + \mu^{23} + \mu^{31}\right)/2$$

Under allocation μ^*, we have $\mu_i^*\left(\left\{\delta_{i/i,i+1}\right\}\right) = \mu_i^*\left(\left\{\delta_{i/i,i-11}\right\}\right) = 1/2$ for all i and $\mu_x^*\left(\{ij\}\right) = 1/2$ for all ij. In words, half of type-i agents are matched with type-$i+1$ agents, and the other half are matched with type-$i-1$ agents, for each agent type i. It is not difficult to verify that μ^* is in the core. Notice that μ^* is not blocked by μ^{12}, μ^{23}, or μ^{31}, as no type-i agent wants to switch to $\mu^{i,i-1}$ under μ^*. Moreover, all other potential blocks include at least one of these three blocks in terms of their support, and therefore they also do not block μ^*. The uniqueness of the core allocation takes some further work. First notice that μ^* is the only allocation under which all agents are matched. Then, it is sufficient to verify that if there is positive mass of type-i agents being unmatched under some allocation μ, then μ is blocked by $\mu^{i,i-1}$. To see this, notice that those unmatched type-i agents are willing to switch to their only bundle $\delta_{i/i,i-1}$ under the block $\mu^{i,i-1}$. Moreover, there is positive mass of type-$i-1$ agents who are willing to switch to their only bundle $\delta_{i-1/i,i-1}$ in block $\mu^{i,i-1}$, as this is the top choice for type-$i-1$ agents, and not all type-$i-1$ agents hold this top choice under allocation μ because a positive mass of type-i agents are unmatched. Therefore, the allocation μ^* is the unique allocation in the core.

As is well known, the finite counterpart of the roommate example above has an empty core. With three agents instead of three types of agents, it becomes a variant of the unstable roommate problem in Gale and Shapley (1962). To see why the core is empty, notice that at least one agent is unmatched under every allocation, and whenever agent i is unmatched, the allocation is blocked by agent i and agent $i-1$.

The example above is a simplistic illustration of the model, as it only involves bilateral contracts and unit-demand agents, where substitutability of preferences holds trivially. The next example, by contrast, involves multilateral contracts, multi-demand agents, imperfectly transferable utility, and complementary preferences, and my aim is to demonstrate the generality of the model.

Example 2 *Consider a board game club in which people meet to play chess and bridge. The bridge game involves four players and the chess game involves two players. There is a continuum of male players and a continuum of female players. Players care about the gender of their opponents in each round of the game they play. When playing chess, the players additionally care about who moves first. When playing bridge, players may gamble, and they value the stake* $s \in [0,1]$. *No player demands more than* 5 *rounds of chess and* 20 *rounds of bridge in one meeting. Each player's preferences are defined over all rounds of*

chess and bridge he/she plays during the meeting. See Appendix 9.2 for a formal description of this example using the notations of the model.

This example involves multilateral contracts, multi-demand agents, imperfectly transferable utility, and possibly complementary preferences. Contracts are multilateral because bridge involves 4 *players, and players have multi-demand because they wish to play multiple rounds. The two players in a chess game can imperfectly transfer their payoffs by transferring the right to move first. Furthermore, players' preferences may exhibit complementarity. For example, male players may insist that at least half of their opponents during a meeting must be female. When this is the case, a male player will reject a round of chess against another male player if this will be his only round of play but is likely to accept it if he has just played two rounds of bridge against three female players. Moreover, because the amount staked* s *on bridge can vary continuously in* $[0,1]$*, this example is outside the scope of Azevedo and Hatfield (2015).*

Now, let us relate the model to the framework of convex matching games introduced in the previous section. A large-economy model with small contracts induces a matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i\in I}\right\}$, where I is the set of agent types and $\mathcal{M}$ is the set of feasible allocations. The characteristic vector of an allocation μ is defined as $\phi(\mu) := (\mu_i(\mathfrak{B}_i)/m_i)_{i\in I}$, i.e., $\phi_i(\mu)$ represents the fraction of type-i agents holding some nonempty bundle under allocation μ. Let $\mathcal{M}_i := \{\mu \in \mathcal{M} : \mu_i(\mathfrak{B}_i) > 0\}$ be the set of allocations that involve a positive mass of type-i agents, and define the domination relation $\sqsupset_i$ from $\mathcal{M}_i$ to $\mathcal{M}$ s.t. $\hat{\mu} \sqsupset_i \mu$ if either $\mu_i(\mathfrak{B}_i) < m_i$ or $\mu_i\left(\left\{\beta \in \mathfrak{B}_i : \hat{\beta} \succ_i \beta\right\}\right) > 0$ for all $\hat{\beta} \in Supp(\hat{\mu}_i)$. In words, $\hat{\mu}$ dominates μ at agent type i if, for every bundle $\hat{\beta}$ that might be held by type-i agents under $\hat{\mu}$, there is positive mass of type-i agents under μ who are willing to switch to $\hat{\beta}$. It is straightforward to verify that the core of the induced matching game G as defined in Definition 3.1 reduces to that defined in Definition 4.1.

Now let us make the crucial observation that the induced matching game has the convex structure defined in Definition 3.2.

Proposition 4.2 *The matching game induced by a large-economy model with small contracts is convex.*

Proof. Let us check the three requirements of a convex matching game.

(1) The allocation space $\mathcal{M}$ is a subset of the vector space

$$\left\{\mu = \left((\mu_i)_{i\in I}\right) : \mu_i \text{ is a finite signed Borel measure over } \mathfrak{B}_i, \text{ for each } i\right\}$$

with addition and scalar multiplication defined component-wise. Clearly, the empty allocation μ^0 is the zero vector.

(2) If $\sum_{j=1}^m w^j \phi(\mu^j) \leq \mathbf{1}$, where $w^j > 0$ and $\mu^j \in \mathcal{M}$ for all $j = 1, 2, \ldots, m$, then the linear combination $\mu := \sum_{j=1}^m w^j \mu^j$ is also a feasible allocation. To see this, note that the total mass constraint

$$\mu_i(\mathfrak{B}_i) = \sum_{j=1}^m w^j \mu_i^j(\mathfrak{B}_i) = m_i \sum_{j=1}^m w^j \phi_i(\mu^j) \leq m_i$$

is satisfied. Moreover, define $\mu_x := \sum_{j=1}^m w^j \mu_x^j$, and the accounting identity $\sum_{i \in I} \int_{\beta \in \mathfrak{B}_i} \beta d\mu_i = \int_{x \in X} x d\mu_x$ holds because it holds for each μ^j and is preserved under the linear combination.

(3) For each $\mu \in \mathcal{M}_i$, let $\underline{\beta}_i(\mu)$ be the worst nonempty bundle for type-i agents under allocation μ, i.e.,

$$\underline{\beta}_i(\mu) := \arg\min_{\succsim_i} Supp(\mu_i)$$

The minimizers exist because $Supp(\mu_i)$ is closed in $\mathfrak{B}_i$, which is compact, and $\succsim_i$ is continuous.[12] Define the relation $\trianglerighteq_i$ over $\mathcal{M}_i$ s.t. $\mu' \trianglerighteq_i \mu$ if $\underline{\beta}_i(\mu') \succsim_i \underline{\beta}_i(\mu)$, where $\underline{\beta}_i(\mu)$ should be viewed as an arbitrary selection from the minimizers, and the relation $\trianglerighteq_i$ defined clearly does not depend on the selection.

Now consider a ϕ_i-convex combination $\mu := \sum_{j=1}^m w^j \mu^j$, i.e., $w^j > 0$ and $\mu^j \in \mathcal{M}$ for all j s.t. $\sum_{j=1}^m w^j \phi_i(\mu^j) = 1$ and $\sum_{j=1}^m w^j \phi(\mu^j) \leq \mathbf{1}$. Further assume that $\mu^j \trianglerighteq_i \hat{\mu}$ for each of its component $\mu^j \in \mathcal{M}_i$. Clearly, μ is a feasible allocation by (2), and it is sufficient to show that $\hat{\mu} \not\trianglerighteq_i \mu$. By definition, we need to show that $\mu_i(\mathfrak{B}_i) = m_i$ and $\mu_i\left(\left\{\beta \in \mathfrak{B}_i : \underline{\beta}_i(\hat{\mu}) \succ_i \beta\right\}\right) = 0$, i.e., all type-$i$ agents are holding some nonempty bundle under allocation μ, and all bundles for type-i agents under μ are weakly more preferred to the worst bundle under block $\hat{\mu}$. First, notice that

$$\mu_i(\mathfrak{B}_i) = \sum_{j=1}^m w^j \mu_i^j(\mathfrak{B}_i) = m_i \sum_{j=1}^m w^j \phi_i(\mu^j) = m_i$$

Second, arbitrarily take a bundle $\beta \in Supp(\mu_i)$. Because

$$Supp(\mu_i) = Supp\left(\sum_{j=1}^m w^j \mu^j\right) = \bigcup_{j:\mu_i^j(\mathfrak{B}_i)>0} Supp(w^j \mu_i^j) = \bigcup_{j:\mu_i^j(\mathfrak{B}_i)>0} Supp(\mu_i^j)$$

[12] The minimizers in $Supp(\mu_i)$ can be obtained by taking the intersection of all lower contour sets within $Supp(\mu_i)$. The intersection is nonempty because $Supp(\mu_i)$ is compact, all lower contour sets are closed, and the intersection of finitely many lower contour sets is nonempty.

we know that there exists j_0 with $\mu^{j_0} \in \mathcal{M}_i$ s.t. $\beta \in Supp(\mu_i^{j_0})$, and thus, $\beta \succsim_i \underline{\beta}_i(\mu^{j_0}) \succsim_i \underline{\beta}_i(\hat{\mu})$, where the second $\succsim_i$ is due to $\mu^{j_0} \trianglerighteq_i \hat{\mu}$. Therefore, the set $\left\{\beta \in \mathfrak{B}_i : \underline{\beta}_i(\hat{\mu}) \succ_i \beta\right\}$ is disjoint with $Supp(\mu_i)$ and thus has zero measure. ■

Intuitively, the convexity of a matching game requires that $\hat{\mu}$ does not dominate a ϕ_i-convex combination of a set of $\trianglerighteq_i$-better allocations at player i. In the large-economy model with small contracts, the relation $\trianglerighteq_i$ is obtained by comparing the worst nonempty bundle under two allocations. To understand how convexity is satisfied by the large-economy model with small contracts, first notice that under the ϕ_i-convex combination, all type-i agents are holding some nonempty bundle since $\sum_{j=1}^{m} w^j \phi_i(\mu^j) = 1$ by definition. Furthermore, every bundle β that is possibly held by type-i agents under the ϕ_i-convex combination must come from some component μ^{j_0} of it that involves type-i agents. By assumption, we have $\mu^{j_0} \trianglerighteq_i \hat{\mu}$, which implies that the bundle β is weakly preferred by type-i agents to the worst bundle under $\hat{\mu}$. As a consequence, under the ϕ_i-convex combination, no type-i agent is willing to switch to the worst bundle under $\hat{\mu}$, and therefore, $\hat{\mu}$ does not dominate μ at agent type i. In particular, notice that the convexity of the induced matching game is satisfied purely because of the structure of the game and is irrelevant to convex preferences.

As an illustration of the convexity requirement and how it is satisfied in this large-economy model, let us consider the continuum roommate example (Example 1). It is straightforward to verify that for each $i = 1$, 2, and 3, the allocation $\mu^* := (\mu^{12} + \mu^{23} + \mu^{31})/2$ is a ϕ_i-convex combination of a set of allocations that are $\trianglerighteq_i$-better than $\mu^{i,i-1}$, as $\mu^{i,i+1} \trianglerighteq_i \mu^{i,i-1}$ and $\sum_{i=1}^{3} \phi(\mu^{i,i+1})/2 = \mathbf{1}$. Then, convexity requires that the allocation $\mu^{i,i-1}$ does not dominate μ^* at player i, i.e., no type-i agent under μ^* is willing to switch to $\mu^{i,i-1}$, which is clearly true according to our previous discussion.

We can also demonstrate the regularity of the matching game G induced by the large-economy model with small contracts.

Proposition 4.3 *The matching game induced by a large-economy model with small contracts is regular.*

Proof. See Appendix 9.3. ■

For the regularity of the induced game, the topology we endow $\mathcal{M}$ with is again the weak-* topology, which in this context is the weakest topology that makes $\int_{\beta \in \mathfrak{B}_i} f d\mu_i$ continuous in μ for each $i \in I$ and each continuous function $f : \mathfrak{B}_i \to \mathbb{R}$. Central to the proof of compactness of $\mathcal{M}$ is the Banach-Alaoglu theorem. The continuity of the domination relation $\sqsupset_i$ is a result of the continuous preference relation $\succsim_i$. The detailed arguments are in Appendix 9.3.

With convexity and regularity, by Theorem 3.4, we have the following nonempty core result.

Theorem 4.4 *In the large-economy model with small contracts, the core is always nonempty.*

This nonempty core result is remarkably general because, essentially, only three assumptions have been made: (1) small contracts, (2) finitely many agent types, and (3) continuous preferences. It does not rely on the two-sided structure of the market, substitutable preferences, or TU, in contrast to the existing results for various core-like solution concepts in the matching literature. Notice that the assumption of small contracts is indispensable. The following example shows that the core may be empty in a continuum economy without assuming small contracts.

Example 3 *Consider a continuum of agents of three types, each of which has mass* 1*. Any set of agents with total mass no greater than* 2 *can form a coalition, and each agent can participate in at most one coalition. Each agent values the mass of each type of agents in the coalition in which he participates. The preferences of a type-i agent are represented by the utility function*

$$u_i(x) = x_i \cdot (9 + 2x_{i+1} + x_{i-1})$$

where $x \in [0,1]^3$ *is the mass vector of the coalition in which he participates. The utility of being alone is* 0*.*

It can be shown that the core is empty in this example, and the detailed arguments are left for Appendix 9.1. As a brief insight, notice that with the preferences specified above, agents of the same type have strong incentives to participate in the same coalition. When all type-i agents participate in one coalition ($x_i = 1$)*, their utility is at least* 9*. However, if, for example, they are equally separated into two coalitions* ($x_i = 1/2$)*, their utility is at most* $1/2 \cdot (9 + 2 + 1) = 6$*. In fact, we can show that all agents of the same type must stay together in the same coalition; otherwise, the allocation is blocked. With this observation, a continuum of three types of agents is equivalent to three discrete agents, and we are essentially back to the unstable roommate problem in Gale and Shapley (1962), where the core is empty.*

The large coalition formation model in the example above fails to induce a convex matching game. To see this, analogous to the continuum roommate example, let us consider three allocations μ^{12}, μ^{23}, and μ^{31}, where, in this context, μ^{ij} stands for the allocation under which all type-i and type-j agents form a large coalition of mass 2, and each agent of the third type stays alone. Consider the linear combination $(\mu^{12} + \mu^{23} + \mu^{31})/2$, which is a ϕ_i-convex combination of a set of allocations that are $\trianglerighteq_i$-better than $\mu^{i,i-1}$, as $\mu^{i,i+1} \trianglerighteq_i \mu^{i,i-1}$ and

$\sum_{i=1}^{3} \phi\left(\mu^{i,i+1}\right)/2 = \mathbf{1}$. Suppose that the matching game induced by the large coalition formation model is convex; then, the linear combination $\left(\mu^{12} + \mu^{23} + \mu^{31}\right)/2$ must correspond to a feasible allocation that is not dominated by $\mu^{i,i-1}$ at agent type i, for each i. However, such an allocation does not exist because, in fact, every allocation in this model is blocked by at least one of μ^{12}, μ^{23}, and μ^{31}, as shown in Appendix 9.1.

5 Finite Economies with Convex Preferences

In this section, I show that the core is nonempty in a finite-economy model with multilateral contracts if all agents have convex and continuous preferences. In this model, the terms of each contract may vary continuously within a convex set, and each agent is assumed to have convex, but not necessarily substitutable, preferences over terms of contracts that involve him. When applied to general equilibrium models, the result reduces to the classic one whereby the core is nonempty given convex preferences and production technologies, but my model is more general because goods are allowed to be agent-specific, which is a common feature shared by existing matching models with contracts (Hatfield and Milgrom (2005)). Moreover, my model does not assume TU and can therefore be applied to markets in which utilities are imperfectly transferable.

Consider an economy with a finite set I of agents who interact through a finite set X of *pre-contracts*. Each pre-contract $x \in X$ involves a nonempty set $I_x \subset I$ of agents and has a set T_x of *contract terms*. In this setup, a *contract* is a pre-contract $x \in X$ paired with a contract term $t_x \in T_x$. Let us assume that for each pre-contract $x \in X$, the set T_x of its terms is a convex and compact subset of a normed vector space. Further, let T_x contain the zero vector 0_x of the normed vector space, which represents the null contract under which the pre-contract x is inactive. For example, a pre-contract type $x \in X$ may be "agent i sells apples to agent j", and we have $I_x = \{i, j\}$. A contract term $t_x \in T_x$ may be $(10, 15)$, which represents "agent i sells 10 lbs. of apples to agent j, and j pays i \$15". Under the null contract term $(0, 0) \in T_x$, agent i sells no apples to agent j and j pays i nothing, and therefore, the pre-contract x is inactive.

For each agent i, let $X_i := \{x \in X : i \in I_x\}$ be the set of pre-contracts that involve agent i, and each agent i only cares about the terms of the pre-contracts in X_i. Formally, each agent i has a complete and transitive preference relation $\succsim_i$ over $T_i := \prod_{x \in X_i} T_x$. Let us assume that agents have convex and upper semi-continuous preferences, in the sense that for each i, the upper contour set $\left\{t_i \in T_i : t_i \succsim_i \hat{t}_i\right\}$ is convex and closed for all $\hat{t}_i \in T_i$.

In this economy, an allocation is $t = (t_x)_{x \in X}$, under which each pre-contract x is assigned

a term $t_x \in T_x$. Let $\bar{T} := \prod_{x \in X} T_x$ be the allocation space. Notice that there is an empty allocation $0 \in \bar{T}$, under which each pre-contract x is assigned its null term 0_x, and therefore, all pre-contracts are inactive. Given an allocation $t \in \bar{T}$, let $I_t := \{i \in I : \exists\, x \in X_i \text{ s.t. } t_x \neq 0_x\}$ be the set of agents involved in some non-null contract under t, and let t_i be the projection of $t \in \prod_{x \in X} T_x$ onto the subspace $\prod_{x \in X_i} T_x$. Let T denote the set of IR allocations, i.e.,

$$T := \left\{ t \in \bar{T} : t_i \succsim_i 0_i \text{ for all } i \right\}$$

Now we are in a position to define blocking and the core. Intuitively, an allocation $\hat{t}$ blocks allocation t if all agents involved in $\hat{t}$ strictly prefer $\hat{t}$ to t.

Definition 5.1 *An IR allocation $t \in T$ is **blocked** by a nonempty IR allocation $\hat{t} \in T \setminus \{0\}$ if $\hat{t}_i \succ_i t_i$ for each $i \in I_{\hat{t}}$. An IR allocation $t \in T$ is in the **core** if it is not blocked by any $\hat{t} \in T \setminus \{0\}$.*

This model is related to Hatfield and Kominers (2015), in which competitive equilibrium, and therefore core allocations, are shown to exist under TU and concave utility functions. In their model, each contract term $t_x \in T_x$ is a combination of a non-monetary term and the monetary transfer for all participants in pre-contract x. All agents' utility functions are quasi-linear in money and concave in the non-monetary term. By contrast, my model does not assume TU and can therefore be applied to markets in which utilities are imperfectly transferable. Another subtle difference is that convexity of preferences in my model corresponds to quasi-concavity of utility functions, which is weaker than the concavity assumed in Hatfield and Kominers (2015).

Now, let us relate the model to the framework of convex matching games introduced in Section 3. The finite-economy model with convex preferences induces a matching game $G = \left\{ I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I} \right\}$ in the following way. Let I be the set of agents and $\mathcal{M} := T$ be the set of IR allocations. The characteristic vector of an allocation $t \in T$ is defined as

$$\phi_i(t) := \begin{cases} 1, \text{ if } i \in I_t \\ 0, \text{ otherwise} \end{cases}$$

for each i, i.e., the characteristic value $\phi_i(t)$ indicates whether agent i is involved in some non-null contract under allocation t. Let $\mathcal{M}_i := \{t \in T : \phi_i(t) = 1\}$ be the set of allocations that involve i agent i, and define the domination relation $\sqsupset_i$ from $\mathcal{M}_i$ to $\mathcal{M}$ s.t. $\hat{t} \sqsupset_i t$ if $\hat{t}_i \succ_i t_i$. In words, allocation $\hat{t}$ dominates t at agent i if agent i under allocation t is willing

to switch to $\hat{t}$. It is straightforward to verify that the core of the induced matching game G as defined in Definition 3.1 reduces to that defined in Definition 5.1.

Now let us make the crucial observation that the induced matching game has the convex structure defined in Definition 3.2. The convexity of the induced matching game is a direct result of convex preferences.

Proposition 5.2 *With convex preferences, the matching game induced by the finite-economy model is convex.*

Proof. Let us check the three requirements of a convex matching game.

(1) Because each T_x is a subset of a vector space, the allocation space $\mathcal{M} := T$ is a subset of the product vector space, with addition and scalar multiplication defined component-wise. Clearly, the empty allocation 0 is the zero vector of the product vector space.

(2) If $\sum_{j=1}^{m} w^j \phi(t^j) \leq \mathbf{1}$, where $w^j > 0$ and $t^j \in \mathcal{M}$ for all $j = 1, 2, \ldots, m$, then the linear combination $t := \sum_{j=1}^{m} w^j t^j$ is also a feasible IR allocation. To see this, for each agent i, we have

$$t_i = \sum_{j=1}^{m} w^j t_i^j = \sum_{j:\phi_i(t^j)=1}^{m} w^j t_i^j \in T_i$$

because $\sum_{j:\phi_i(t^j)=1}^{m} w^j = \sum_{j=1}^{m} w^j \phi_i(t^j) \leq 1$, and therefore, t is a feasible allocation. Furthermore, because $t_i^j \succsim_i 0_i$ for each j with $\phi_i(t^j) = 1$, by the convexity of $\succsim_i$, we have $t_i \succsim_i 0_i$. Therefore, allocation t is IR.

(3) Define the relation $\trianglerighteq_i$ over $\mathcal{M}_i$ s.t. $t' \trianglerighteq_i t$ if $t'_i \succsim_i t_i$. Consider a ϕ_i-convex combination $t := \sum_{j=1}^{m} w^j t^j$, i.e., $w^j > 0$ and $t^j \in \mathcal{M}$ for all j s.t. $\sum_{j=1}^{m} w^j \phi_i(t^j) = 1$ and $\sum_{j=1}^{m} w^j \phi(t^j) \leq \mathbf{1}$. Further assume that $t^j \trianglerighteq_i \hat{t}$ for each of its components $t^j \in \mathcal{M}_i$. Clearly, t is a feasible IR allocation by (2), and it is sufficient to show that $\hat{t} \ntriangleright_i t$. However, this is trivial because

$$t_i = \sum_{j=1}^{m} w^j t_i^j = \sum_{j:t^j \in \mathcal{M}_i}^{m} w^j t_i^j \succsim_i \hat{t}_i$$

because $\sum_{j:t^j \in \mathcal{M}_i}^{m} w^j = \sum_{j=1}^{m} w^j \phi_i(t^j) = 1$ and $\succsim_i$ is convex. ■

We can also demonstrate the regularity of the matching game G induced by the finite-economy model given upper semi-continuous preferences.

Proposition 5.3 *With upper semi-continuous preferences, the matching game induced by the finite-economy model is regular.*

Proof. (1) Compactness of T.

The space T of IR allocations can be represented as

$$T := \bigcap_{i \in I} \left\{ t \in \bar{T} : t_i \succsim_i 0_i \right\}$$

Because $\bar{T} := \prod_{x \in X} T_x$ and each T_x is compact, we know that $\bar{T}$ is compact. By the upper semi-continuity of $\succsim_i$, the set $\left\{ t \in \bar{T} : t_i \succsim_i 0_i \right\}$ is closed in $\bar{T}$. Thus, T is closed in $\bar{T}$ and therefore compact.

(2) Continuity of $\sqsupset_i$.

The set $\left\{ t \in T : \hat{t} \not\sqsupset_i t \right\} = \left\{ t \in T : t_i \succsim_i \hat{t}_i \right\}$ is closed by the upper semi-continuity of $\succsim_i$.

■

With convexity and regularity, by Theorem 3.4, we have the following nonempty core result.

Theorem 5.4 *With convex and upper semi-continuous preferences, the finite economy has a nonempty core.*

6 Large-firm Matching with Peer Preferences

In this section, I study a large-firm many-to-one matching model with peer preferences. There are finitely many firms and a continuum of workers in the market, and each firm is large in the sense that it can hire a continuum of workers. Workers may have preferences over their peers, in the sense that they value not only the firm for which they work but also the colleagues with whom they work. With peer preferences, each contract in this model is multilateral and involves a firm and the set of all workers the firm employs. Notice that the nonempty core result in Section 4 does not apply because contracts are large since firms may hire a continuum of workers with positive mass.

In this model, I show that the core is nonempty if all firms and workers have convex and continuous preferences and, in addition, all workers' preferences over peers satisfy a "competition aversion" condition. Roughly speaking, the competition-aversion condition requires that each worker does not like colleagues of his own type, possibly because workers of the same type have to compete for projects, resources, and promotions when employed by the same firm. A striking observation is that the core may be empty without the competition-aversion condition, even if all firms and workers have convex and continuous preferences. In other words, convexity of preferences is not sufficient for convexity of the induced matching game, in contrast to the model in the last section. As we will see, convexity of the

induced matching game is only satisfied when convexity of preferences is combined with the competition-aversion condition.

Consider a finite set F of firms and a continuum of workers of finitely many types. Let Θ be the set of worker types, and for each worker type $\theta \in \Theta$, let $m(\theta) > 0$ be the mass of type-θ workers in the market. Then, a set of workers can be represented by a non-negative mass vector $x \in \mathbb{R}_+^{\Theta}$ subject to the constraint $x(\theta) \leq m(\theta)$. Let X be the set of all such mass vectors. An allocation is defined as $\mu = (\mu_f)_{f \in F}$, where $\mu_f \in X$ is the mass vector representing the set of workers employed by firm f. For the matching μ to be feasible, it has to respect the total mass of workers of each type, i.e., $\sum_{f \in F} \mu_f \leq m$.

The firms' preferences are defined over sets of workers. Formally, each firm f has a complete and transitive preference relation $\succsim_f$ over X that is upper semi-continuous and convex, i.e., the upper contour set $\{x \in X : x \succsim_f x_0\}$ is closed and convex for each $x_0 \in X$. Workers value not only the firm for which they work but also the colleagues with whom they work, and therefore, each worker has a complete and transitive preference relation $\succsim_\theta$ over $(F \times X) \cup \{\emptyset\}$. The pair $(f, x) \in F \times X$ represents the state of being employed by firm f, the whole set of f's employees is represented by the mass vector x, and the symbol $\emptyset$ represents the state of being unemployed. Let us assume each worker type θ's preference relation $\succsim_\theta$ to be upper semi-continuous and convex, i.e., the upper contour set $\{x \in X : (f, x) \succsim_\theta a\}$ is closed and convex, for each alternative $a \in (C \times X_\theta) \cup \{\emptyset\}$ and each firm $f \in F$.

We say that a match $(f, x) \in F \times X$ is IR if firm f and the workers involved in x find the match acceptable, i.e., $x \succsim_f 0$ and $(f, x) \succsim_\theta \emptyset$ for each worker type θ with $x(\theta) > 0$. A feasible matching μ is said to be IR if (f, μ_f) is IR for all firms $f \in F$. Let $\mathcal{M}$ be the set of all feasible and IR matchings. Notice that there is an empty matching $\mu^0 \in \mathcal{M}$ under which all workers are unemployed, i.e., $\mu_f^0 = 0$ for all $f \in F$. In the remainder of this section, a matching always refers to a feasible and IR matching, unless otherwise stated.

Now let us define the notions of blocking and the core.

Definition 6.1 *A matching $\mu \in \mathcal{M}$ is **blocked** by a nonempty matching $\mu \in \mathcal{M} \setminus \{\mu^0\}$ if for each $f \in F$ with $\mu_f \neq 0$, we have $\hat{\mu}_f \succ_f \mu_f$ and for each $\hat{f}$ and θ with $\hat{\mu}_{\hat{f}}(\theta) > 0$, we have either $\sum_{f \in F} \mu_f(\theta) < m(\theta)$ or there exists $f \in F$ with $\mu_f(\theta) > 0$ s.t. $(\hat{f}, \hat{\mu}_{\hat{f}}) \succ_\theta (f, \mu_f)$. A matching $\mu \in \mathcal{M}$ is in the **core** if it is not blocked by a nonempty matching.*

Intuitively, a matching $\hat{\mu}$ blocks an another matching μ if all firms involved in the block $\hat{\mu}$ are willing to switch to $\hat{\mu}$ from μ, and for each worker type θ that is employed by firm $\hat{f}$ under the block $\hat{\mu}$, we have to find a positive mass of type-θ workers under μ who are willing to switch to $(\hat{f}, \hat{\mu}_{\hat{f}})$. These type-$\theta$ workers may come from two sources: workers are

unemployed under μ and workers who are employed by some firm f under μ, but strictly prefer $(\hat{f}, \hat{\mu}_{\hat{f}})$ to (f, μ_f).

This model is related to the recent paper by Che, Kim, and Kojima (2017), who show that stable matchings always exist in a large-firm many-to-one matching model without peer preferences, provided that all firms have continuous and convex, but not necessarily substitutable, preferences. Without peer preferences, my notion of the core is slightly different from the notion of stable matchings in Che, Kim, and Kojima (2017). Moreover, their paper allows for a compact space of worker types, but I only consider finitely many types of workers.

Surprisingly, the assumptions of convexity and upper semi-continuity I have imposed on preferences thus far are not sufficient for a nonempty core. The following provides a counter-example.

Example 4 *Consider two firms and three worker types θ_1, θ_2, and θ_3, and each type of worker has a mass of 1. All firms strictly prefer to have more workers but have a capacity constraint of mass 2. For each worker type θ_i, all type-θ_i workers' preferences over $F \times X$ are represented by the utility function*

$$u_i(f, x) = x_i \cdot (9 + 2x_{i+1} + x_{i-1})$$

Because the two firms are dummy agents that passively accept workers, this example reduces to the large coalition formation example (Example 3), in which the core is empty.

As in Example 3, the crucial observation from the example above is that if workers of the same type have a strong incentive to stay together by working for the same firm, we are essentially back to a finite matching market with three discrete workers. With dummy firms and preferences over peers, the model essentially becomes a finite roommate problem, in which the core may very well be empty. Therefore, to obtain a nonempty core, some assumption has to be made such that agents of the same type are willing to be separated into different firms. Following the observation above, I find that the following additional restriction on workers' preferences is sufficient for a nonempty core.

Definition 6.2 *The workers of type $\theta \in \Theta$ are competition-averse if $(f, x') \succsim_\theta (f, x)$ for every pair of IR matches with $x(\theta) > 0$ and $x'(\theta) = 0$.*

The competition aversion restriction defined above requires that a type-θ worker approaches his bliss point when the mass of type-θ colleagues approaches 0. This implies that the worker does not like colleagues of the same type, possibly due to competition for

projects, resources, and promotions among the same type of workers employed by the same firm. Admittedly, in applications where there is strong synergy between workers of the same type, the competition-aversion condition is not satisfied, and the core is not guaranteed to be nonempty. As Example 3demonstrates, if same-type synergy is sufficiently strong that all workers of the same type always wish to stay together in one firm, we are back to a finite coalition formation problem where the core is often empty. However, if the same-type synergy is dominated by the competition effect, then the competition-aversion condition is a reasonable assumption, in which case the core is guaranteed to be nonempty.

Now, let us relate the model to the framework of convex matching games introduced in Section 3. The large-firm matching model with peer preferences induces a matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$ in the following way. Let $I := F \cup \Theta$, and $\mathcal{M}$ be the set of feasible IR matchings. The characteristic vector of an allocation $\mu \in \mathcal{M}$ is defined as

$$\phi_f(\mu) := \begin{cases} 1, \text{ if } \mu_f \neq 0 \\ 0, \text{ otherwise} \end{cases}, \text{ for each } f \in F$$

$$\phi_\theta(\mu) := \frac{\sum_{f \in F} \mu_f(\theta)}{m(\theta)}, \text{ for each } \theta \in \Theta$$

i.e., the characteristic value $\phi_f(\mu)$ indicates whether firm ϕ is involved in matching μ, and $\phi_\theta(\mu)$ represents the fraction of type-θ workers employed under matching μ. Let $\mathcal{M}_i := \{t \in T : \phi_i(t) > 0\}$ for each $i \in F \cup \Theta$, and define the domination relation $\sqsupset_i$ from $\mathcal{M}_i$ to $\mathcal{M}$. For a firm f, let $\hat{\mu} \sqsupset_f \mu$ if $\hat{\mu}_f \succ_f \mu_f$, i.e., firm f prefers the set of employees under $\hat{\mu}$ to μ. For a worker of type θ, let $\hat{\mu} \sqsupset_\theta \mu$ if either $\sum_{f \in F} \mu_f(\theta) < m(\theta)$ or for each $\hat{f}$ with $\hat{\mu}_{\hat{f}}(\theta) > 0$ there exists $f \in F$ with $\mu_f(\theta) > 0$ s.t. $(\hat{f}, \hat{\mu}_{\hat{f}}) \succ_\theta (f, \mu_f)$. In words, a matching $\hat{\mu}$ dominates a matching μ at worker type θ if there exists a positive mass of type-θ workers under matching μ willing to switch to any position for type-θ workers in matching $\hat{\mu}$. It is straightforward to verify that the core of the induced matching game G as defined in Definition 3.1 reduces to that defined in Definition 6.1.

Now let us make the crucial observation that the induced matching game has the convex structure defined in Definition 3.2. The convexity of the induced matching game is a result of convex preferences and competition aversion.

Proposition 6.3 *If all firms and workers have convex preferences and all workers are competition-averse, the matching game induced by the large-firm matching model with peer preferences is convex.*

Proof. See Appendix 9.4. ■

Intuitively, the convexity of a matching game requires that for each i, an allocation $\hat{\mu} \in \mathcal{M}_i$ does not dominate a ϕ_i-convex combination of a set of $\trianglerighteq_i$-better allocations at player i. In this model, the convexity of preferences alone is insufficient for the convexity of the induced matching game. This is because the relation $\trianglerighteq_\theta$ is only concerned with the firms that employ a positive mass of type-θ agents, but the components of a ϕ_θ-convex combination may involve firms that do not employ type-θ workers. Therefore, to ensure that the ϕ_θ-convex combination is unblocked, we need to require that a mass vector x with $x(\theta) = 0$ is preferred to mass vectors with $x(\theta) > 0$. In this large-firm model with peer preferences, a notable difference from the first application in this paper is that a component of a ϕ_θ-convex combination with $\phi_\theta(\mu) = 0$ may affect type θ workers' payoff, while in a large economy with small contracts, a component with $\phi_i(\mu) = 0$ is not relevant to type-i agents.

Moreover, we can demonstrate the regularity of the matching game G induced by the large-firm matching model, given upper semi-continuous preferences.

Proposition 6.4 *If all firms and workers have upper semi-continuous preferences, the matching game induced by the large-firm matching model with peer preferences is regular.*

Proof. (1) Compactness of $\mathcal{M}$.

Let $\bar{\mathcal{M}}$ be the set of feasible but not necessarily IR matchings, i.e.,

$$\bar{\mathcal{M}} := \left\{ \mu = (\mu_f)_{f \in F} : \mu_f \in \mathbb{R}_+^\Theta \text{ and } \sum_{f \in F} \mu_f \leq m \right\}$$

Clearly, $\bar{\mathcal{M}}$ is compact (w.r.t. Euclidean metric). Then, the space $\mathcal{M}$ of feasible IR matchings can be represented as $\mathcal{M} = \bigcap_{f \in F} \mathcal{M}_f \cap \bigcap_{\theta \in \Theta} \mathcal{M}_\theta$, where

$$\begin{aligned} \mathcal{M}_f &:= \left\{ \mu \in \bar{\mathcal{M}} : \mu_f \succsim_f 0 \right\} \\ \mathcal{M}_\theta &:= \left\{ \mu \in \bar{\mathcal{M}} : (f, \mu_f) \succsim_\theta \emptyset \text{ for each } f \text{ with } \mu_f(\theta) > 0 \right\} \end{aligned}$$

The set $\mathcal{M}_f$ is closed by upper the semi-continuity of $\succsim_f$, and the set $\mathcal{M}_\theta$ is closed by the upper semi-continuity of $\succsim_\theta$. Therefore, $\mathcal{M}$ is closed in $\bar{\mathcal{M}}$ and thus compact.

(2) Continuity of $\sqsupset_i$.

For each firm f, the set

$$\{\mu \in \mathcal{M} : \hat{\mu} \not\sqsupset_f \mu\} = \{\mu \in \mathcal{M} : \mu_f \succsim_f \hat{\mu}_f\}$$

is closed by the upper semi-continuity of $\succsim_f$.

For each worker type θ, the set $\{\mu \in \mathcal{M} : \hat{\mu} \not\supseteq_f \mu\}$ can be represented as

$$\begin{aligned}&\left\{\mu \in \mathcal{M} : \textstyle\sum_{f \in F} \mu_f(\theta) = m(\theta)\right\} \\ &\cap \bigcup_{\hat{f} \in F : \hat{\mu}_{\hat{f}}(\theta) > 0} \left\{\mu \in \mathcal{M} : (f, \mu_f) \succsim_\theta (\hat{f}, \hat{\mu}_{\hat{f}}) \text{ for each } f \text{ with } \mu_f(\theta) > 0\right\}\end{aligned}$$

Clearly, the set $\left\{\mu \in \mathcal{M} : \sum_{f \in F} \mu_f(\theta) = m(\theta)\right\}$ is closed. Moreover, the set

$$\left\{\mu \in \mathcal{M} : (f, \mu_f) \succsim_\theta (\hat{f}, \hat{\mu}_{\hat{f}}) \text{ for each } f \text{ with } \mu_f(\theta) > 0\right\}$$

is closed by the upper semi-continuity of $\succsim_f$. ■

With convexity and regularity, by Theorem 3.4, we have the following nonempty core result.

Theorem 6.5 *In the large-firm matching model with peer preferences, the core is nonempty if all firms and workers have upper semi-continuous and convex preferences and all workers are competition-averse.*

The model subsumes the case in which workers have no peer preferences by letting $(f, x') \sim_\theta (f, x)$ for all x, $x' \in X$. Thus, the next corollary is immediate.

Corollary 6.6 *In the large-firm matching model without peer preferences, the core is nonempty if all firms and workers have upper semi-continuous and convex preferences.*

The corollary above does not subsume the main result of Che, Kim, and Kojima (2017), as their notion of stable matchings is slightly different from my notion of the core. Furthermore, their model allows for a compact space of worker types, but I only consider finitely many types of workers.

7 Scarf's Lemma Approach to Core

In the previous three sections, each model induces a regular and convex matching game, which is claimed to have a nonempty core by Theorem 3.4. This section is devoted to the proof of Theorem 3.4 using Scarf's lemma.

Scarf's lemma first appeared in the seminal paper Scarf (1967). Consider an n-by-m non-negative matrix $A = (a_{i,j})$, each of whose columns has at least one positive entry. Each row i of A is associated with a complete and transitive relation $\trianglerighteq_i$ over those column js with $a_{i,j} > 0$. It is said that a vector w in the polyhedron $\left\{w \in \mathbb{R}^m_+ : Aw \leq \mathbf{1}\right\}$ *dominates column*

j at row i if $a_{i,j} > 0$, $\sum_{k=1}^{m} w^k a_{i,k} = 1$, and $k \trianglerighteq_i j$ for all k with $w^k > 0$ and $a_{i,k} > 0$. In words, a vector w dominates column j at row i if the inequality $Aw \leq \mathbf{1}$ is binding at row i, j is in the domain of $\trianglerighteq_i$, and $\trianglerighteq_i$-dominated by every column k in the support of w, provided that k is also in the domain of $\trianglerighteq_i$. We also say that a vector w *dominates column j*, without specifying at which row, if it does so at some row.

Lemma 7.1 (Scarf, 1967) *There exists a vector of the polyhedron $\left\{w \in \mathbb{R}_+^m : Aw \leq \mathbf{1}\right\}$ that dominates every column of A.*[13]

To see how Scarf's lemma is related to the core of a convex matching game, consider a convex matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$. Each finite family $(\mu^j)_{j=1}^m$ of nonempty allocations induces a matrix A, whose rows are indexed by I and columns are indexed by $j \in \{1, 2, \ldots, m\}$. The j-th column of A is the characteristic vector $\phi(\mu^j)$ of the j-th allocation μ^j. Because μ^j is nonempty, its characteristic vector $\phi(\mu^j) \neq \mathbf{0}$. Let the $\trianglerighteq_i$ associated with row i be the relation guaranteed to exist by Definition 3.2(3). Then, the following observation is straightforward.

Proposition 7.2 *Given a matrix A induced by a finite family $(\mu^j)_{j=1}^m$ of nonempty allocations in a convex matching game, if a vector $w \in \mathbb{R}_+^m$ with $Aw \leq \mathbf{1}$ dominates column j at row i, then the w-linear combination $\mu := \sum_{j=1}^m w^j \mu^j$ is also an allocation in $\mathcal{M}$, and furthermore, $\mu^j \not\trianglelefteq_i \mu$.*

Proof. The linear combination $\mu = \sum_{j:w_j>0} w^j \mu^j$ is in the allocation space $\mathcal{M}$ by Definition 3.2(2), as $\sum_{j:w_j>0} w^j \phi(\mu^j) = Aw \leq \mathbf{1}$. We also have $\mu^j \not\trianglelefteq_i \mu$ by Definition 3.2(3), as $\sum_{j:w_j>0} w^j \phi_i(\mu^j) = 1$ and $\mu^k \trianglerighteq_i \mu^j$ for all k with $w^k > 0$ and $\mu^j \in \mathcal{M}_i$. ■

Scarf's lemma claims that there exists a vector $w^* \in \mathbb{R}_+^m$ with $Aw^* \leq \mathbf{1}$ that dominates every column of A. Then, the w^*-linear combination $\mu^* := \sum_{j=1}^m w^{*j} \mu^j$ is an allocation that is not blocked by any μ^j, as $\mu^j \not\trianglelefteq_i \mu^*$ by the Proposition above. Then, we have the following corollary.

Corollary 7.3 *Given a finite family $(\mu^j)_{j=1}^m$ of nonempty allocations in a convex matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$, there exists allocation $\mu^* \in \mathcal{M}$ that is not blocked by each μ^j in this finite family.*

From the corollary above, we know that we can find an allocation μ^* that is not blocked by finitely many allocations. However, to obtain an allocation in the core, in general, we have

[13] This formulation can be found, for example, in Kiraly and Pap (2008). In the combinatorial literature, the relation $\geq_i$ is often assumed to be a total order. However, in fact, we can relax anti-symmetry to allow ties.

to rule out infinitely many allocations since each nonempty allocation is a potential block. This gap between finitely and infinitely many blocks is filled by the regularity condition.

Let $\mathcal{M}^* \subset \mathcal{M}$ be the core of a regular and convex matching game G. By definition, it can be represented as

$$\mathcal{M}^* = \bigcap_{\hat{\mu} \in \mathcal{M} \setminus \{\mu^0\}} \mathcal{M}^*_{\hat{\mu}}$$

where $\mathcal{M}^*_{\hat{\mu}} \subset \mathcal{M}$ represents the set of allocations that cannot be blocked by $\hat{\mu}$. It is well known that in a compact topological space, a family of closed sets has a nonempty intersection if every finite sub-family has a nonempty intersection. By the regularity of the matching game G, the allocation space $\mathcal{M}$ is compact. Each $\mathcal{M}^*_{\hat{\mu}}$ is closed in $\mathcal{M}$ by the continuity of $\sqsupset_i$ because, by definition,

$$\mathcal{M}^*_{\hat{\mu}} = \bigcup_{i \in I : \phi_i(\hat{\mu}) > 0} \{\mu \in \mathcal{M} : \hat{\mu} \not\sqsupset_i \mu\}$$

Finally, Corollary 7.3 implies that the intersection of every finite family of $\mathcal{M}^*_{\hat{\mu}}$s is nonempty, and therefore, the core $\mathcal{M}^*$, as the intersection of all $\mathcal{M}^*_{\hat{\mu}}$s, is also nonempty. This completes the proof of the central result of this paper, Theorem 3.4.

To better understand how the Scarf's lemma approach to the core works, it is helpful to illustrate it in the continuum roommate example (Example 1). Take three nonempty allocations, where μ^{ij} represents the allocation under which each type-i agent is matched with a type-j agent, and all agents of the third type are unmatched. I will illustrate how Scarf's lemma provides allocation that is not blocked by any of these three allocations.

These three allocations μ^{12}, μ^{23}, and μ^{31} induce the following matrix A, whose rows are indexed by agent types and columns are indexed by the three allocations. For each row $i \in \{1, 2, 3\}$ and column $\mu \in \{\mu^{12}, \mu^{23}, \mu^{31}\}$, the (i, μ)-th entry is the fraction of matched type-i agents under μ. Thus, matrix A is

	μ^{12}	μ^{23}	μ^{31}
Type 1	**1**	0	1
Type 2	1	**1**	0
Type 3	0	1	**1**

For each row i, we define $\mu^{i,i+1} \trianglerighteq_i \mu^{i,i-1}$ since type-i agents prefer a type-$i+1$ roommate to a type-$i-1$ roommate. I use **bold 1** to indicate the more preferred allocation for each row.

Scarf's lemma asserts that there exists a vector in the polyhedron $\left\{w \in \mathbb{R}^3_+ : Aw \leq \mathbf{1}\right\}$ that dominates all columns. In this example, it is not difficult to verify that this dominating

vector is unique, which is $w^* := (1/2, 1/2, 1/2)$. Then, allocation μ^* is constructed as the w^*-linear combination of the three allocations

$$\mu^* := \frac{1}{2}\mu^{12} + \frac{1}{2}\mu^{23} + \frac{1}{2}\mu^{31}$$

Under allocation μ^*, for each type i, half of the type-i agents are matched to a type-$i+1$ roommate, and the other half are matched to a type-$i-1$ roommate. As we have already seen in Section 4, the allocation μ^* is not blocked by μ^{12}, μ^{23}, or μ^{31}. In fact, it is straightforward to verify that w^* dominates column $\mu^{i,i+1}$ at row $i+1$ for each i. By Proposition 7.2, this implies that the allocation $\mu^{i,i+1}$ does not block μ^* because no type-$i+1$ agent is willing to participate.

The convexity of the matching game plays an important role in the Scarf's lemma approach to the core. After obtaining a dominating vector w^* using Scarf's lemma, we need to use convexity to ensure that the w^*-linear combination $\mu^* := \sum_{j=1}^{m} w^{*j}\mu^j$ is an allocation that is not blocked by any μ^j. The existence of this μ^* is not guaranteed without the convexity of the matching game. For example, consider the large coalition formation example (Example 3) in Section 4. Consider the three allocations μ^{12}, μ^{23}, and μ^{31}, where, in this context, μ^{ij} stands for the allocation under which all type-i and type-j agents form a large coalition of mass 2, and each agent of the third type remains alone. The induced matrix A is the same as that in the continuum roommate example, and we will obtain the same unique dominating vector $w^* = (1/2, 1/2, 1/2)$. However, without convexity, we cannot convert this dominating vector into an unblocked allocation μ^*. In fact, such an allocation does not exist, as it can be shown that every allocation in this model is blocked by at least one of μ^{12}, μ^{23}, and μ^{31}. See Appendix 9.1 for detailed arguments.

8 Conclusion

In this paper, I show that the core is nonempty in all matching games that satisfy a convexity structure, including a large class of models that allow for arbitrary contracting networks, multilateral contracts, and complementary preferences. Roughly speaking, the convexity of a matching game requires that the allocation space is a convex set and that if a player weakly prefers a set of allocations to a potential block, then the player is unwilling to participate in the block under any convex combination of these allocations. Scarf's lemma is central to my approach to the nonemptiness of the core, in contrast to the standard fixed-point approach in matching theory.

Three applications of the framework of convex matching games are provided, and convex-

ity is satisfied by a different set of assumptions in each application. In the first application, I consider a continuum large-economy model with small contracts, in the sense that the set of participants in each contract has zero mass. Given finitely many types of agents and continuous preferences, I show that the core is always nonempty even with arbitrary contracting networks, multilateral contracts, and arbitrary, possibly complementary, preferences. In this application, the convexity of the matching game is satisfied because of the assumptions of small contracts, and the convexity of preferences is not relevant. In the second application, I show that the core is nonempty in a finite-economy model with multilateral contracts if all agents have convex and continuous preferences. The convexity in this application is satisfied in a straightforward way because of convex preferences. In the third application, I study a large-firm, many-to-one matching model with peer preferences, and I show that the core is nonempty if all firms and workers have convex and continuous preferences and, in addition, all workers are competition-averse. In this application, the convexity of the matching game is satisfied by the convexity of preferences combined with competition aversion.

One might be curious about the maximum scope of the Scarf's lemma approach developed in this paper. In Appendix 9.6, I explore this issue and provide a more general structure to which the Scarf's lemma approach can be applied. This more general structure for matching games is a balancedness structure that generalizes the pivotal balancedness condition introduced by Iehle (2007). I show that the generalized pivotal balancedness structure is necessary and sufficient for a nonempty core.

As a final remark, the framework of convex matching games introduced in this paper may also be applied to proving the existence of other core-like solution concepts, such as stability. When we apply the general framework to a specific model, we have some freedom to use different versions of the domination relations $(\sqsupset_i)_{i\in I}$, in which case the core of the matching game may correspond to different core-like solution concepts. As long as the induced matching game is convex, we can use the Scarf's lemma approach to demonstrate the nonemptiness of the core, which implies the existence of allocations that satisfy those alternative solution concepts. For example, in Appendix 9.5, I use the Scarf's lemma approach developed in this paper to show a special case of a stability result in Azevedo and Hatfield (2015).

9 Appendix

9.1 Empty Core in Example 3

I provide the formal arguments for empty core in Example 3. In fact, we can show that every allocation is blocked by at least one of these three allocations: μ^{12}, μ^{23}, and μ^{31}, where

μ^{ij} stands for the allocation under which all type i and type j agents form a large coalition of mass 2, and each agent of the third type stays alone.

To see this, suppose that there is an allocation μ that is not blocked by any of these three blocks. An type i agent's utility is at most 11, which is only achieved under allocation $\mu^{i,i+1}$. However, $\mu^{i,i+1}$ is blocked by $\mu^{i+1,i-1}$, and therefore all agents' utility must be strictly less than 11 under allocation μ. As a consequence, all type i agents are willing to participate in the block $\mu^{i,i+1}$.

For each i, because $\mu^{i-1,i}$ does not block μ while all type $i-1$ agents are willing to participate, it must be the case that there is some type i agents who are unwilling to participate in $\mu^{i-1,i}$. This can only happen when their utility is weakly greater than 10 under μ. This implies that these type i agents are in some coalition x^i with $x^i_i > 5/6$. Suppose that the coalition x^i is not the same coalition as x^{i+1} or x^{i-1}, then we have $x^i_{i+1} < 1/6$ and $x^i_{i-1} < 1/6$, which contradicts $x^i_i \cdot \left(9 + 2x^i_{i+1} + x^i_{i-1}\right) > 10$. Therefore, the three coalitions x^1, x^2, and x^3 must be the same coalition x. However, $x_i > 5/6$ for each i contradicts the maximum capacity of a coalition. This concludes the proof that every allocation is blocked by at least one of the three blocks μ^{12}, μ^{23}, and μ^{31}.

9.2 Example 2 Continued

In the terminologies of the model, there are two types of agents, $I = \{m, f\}$. The set of roles is

$$\begin{aligned} R = &\left\{c^i_{j,l} : i, j \in I,\ l = 1, 2\right\} \\ &\cup \left\{b^i_{k,s} : k = 0, 1, 2, 3,\ s \in [0, 1]\right\} \end{aligned}$$

where $c^i_{j,l}$ represents "playing one round of chess as a gender i player against a gender j player, and being the l-th mover", and b^i_k represents "playing one round of bridge as a gender i player, where k of my opponents are male". The set of contract types is

$$X = \{c_0, c_{1,f}, c_{1,m}, c_2\} \cup \{b_{k,s} : k = 0, 1, 2, 3, 4,\ s \in [0, 1]\}$$

where the contract type c_2 (or c_0) represents a round of chess with 2 male (or female) players, the contract type $c_{1,f}$ (or $c_{1,m}$) represents a round of chess with a male and a female player where the female (or male) player is the first mover, and the contract type $b_{k,s}$ represents a round of bridge with k male players and $4-k$ female players and stake s. All contract types

in X are measures over R, which are specified as follows:

$$\begin{aligned} c_0 &= \delta[c^f_{f,1}] + \delta[c^f_{f,2}] \\ c_{1,f} &= \delta[c^f_{m,1}] + \delta[c^m_{f,2}] \\ c_{1,m} &= \delta[c^f_{m,2}] + \delta[c^m_{f,1}] \\ c_2 &= \delta[c^m_{m,1}] + \delta[c^m_{m,2}] \\ b_{k,s} &= k \cdot \delta[b^m_{k-1,s}] + (4-k) \cdot \delta[b^f_{3-k,s}] \end{aligned}$$

for $k = 0, 1, 2, 3, 4$ and $s \in [0, 1]$, where $\delta[r]$ is the degenerate measure that assigns measure 1 to r and measure 0 elsewhere.

A bundle β of roles is an integer-valued measure over R that assigns measure 1 to at most 5 chess roles and 20 bridge roles, possibly duplicate. Then allocations and the core are defined in a straightforward way.

9.3 Technical Notes on Large Economies with Small Contracts

In this appendix, I'm going to repeatedly use the following mathematical fact.

Lemma 9.1 *Let K be a compact metric space, and h be a positive real number. Then the set of Borel measures μ over K with $\mu(K) \leq h$, endowed with the weak-* topology, is compact and metrizable.*

Proof. By Banach-Alaoglu theorem. [To be elaborated] ■

The next corollary is immediate, which states that compactness w.r.t. weak-* topology is no more than sequential closedness and boundedness.

Corollary 9.2 *A set $\mathcal{M}$ of Borel measures over a compact metric space K is compact w.r.t. the weak-* topology, iff $\mathcal{M}$ is sequentially closed and there exists $h > 0$ s.t. $\mu(K) \leq h$ for all $\mu \in \mathcal{M}$.*

Proof. "If":

Let $\bar{\mathcal{M}}$ be the set of all Borel measures μ over K with $\mu(K) \leq h$. By the previous lemma, $\bar{\mathcal{M}}$ is compact and metrizable. Because $\mu(K)$ is continuous in μ, $\mathcal{M}$ is sequentially closed in $\bar{\mathcal{M}}$. Then by metrizability, $\mathcal{M}$ is closed in $\bar{\mathcal{M}}$, and therefore compact.

"Only if":

Because $\mu(K)$ is continuous in μ, and $\mathcal{M}$ is compact, the image $\{\mu(K) : \mu \in \mathcal{M}\}$ is compact in $\mathbb{R}$. Therefore, there exists $h > 0$ s.t. $\mu(K) \leq h$ for all $\mu \in \mathcal{M}$. Then set $\mathcal{M}$ is a

subset of $\bar{\mathcal{M}}$, which is metrizable by the previous lemma. Then compactness of $\mathcal{M}$ implies closedness, which is equivalent to sequential closedness. ■

In the large-economy model with small contracts, the set $\mathfrak{B}$ of bundles is defined as

$$\mathfrak{B} := \left\{ \sum_{n=1}^{N'} \delta_{r_n} : r_n \in R \text{ for each } n,\ N' \leq N \right\}$$

where δ_r is the Dirac measure.

Proposition 9.3 *The set $\mathfrak{B}$ of bundles is compact.*

Proof. Let's show that the set $\mathfrak{B}$ is sequentially closed in the compact set

$$\bar{\mathfrak{B}} := \{\text{Borel measure } \beta \text{ over } R : \|\beta\| \leq N\}$$

which is going to imply compactness of $\mathfrak{B}$. Take any sequence $\left(\beta^l\right)$ in $\mathfrak{B}$ convergent to β^0 in $\bar{\mathfrak{B}}$. I want to show that β^0 is also in $\mathfrak{B}$. Let $\beta^l = \sum_{k=1}^{n_l} \delta_{r_k^l}$. Because $\beta^l(R) = n_l$ is convergent, we know that $n_l = n$ when l is large enough. Consider the sequence $\left(r_1^l\right)$, take a subsequence convergent to r_1. Take the indices l in the subsequence, and consider the sequence $\left(r_2^l\right)$, find a subsequence convergent to r_2. Repeat this process, we find a subsequence $\beta^l = \sum_{k=1}^{n} \delta_{r_k^l}$ s.t. $r_k^l \to r_k$ for each k. Then we have $\delta_{r_k^l} \overset{w^*}{\to} \delta_{r_k}$ and so $\beta^l \overset{w^*}{\to} \beta^* := \sum_{k=1}^{n} \delta_{r_k}$ in the subsequence. Because the whole sequence converges to β^0, we know that $\beta^0 = \beta^*$, which is in $\mathfrak{B}$. ■

Let $\mathfrak{B}_i$ be the set if nonempty IR bundles for type i agents, i.e.

$$\mathfrak{B}_i := \{\beta \in \mathfrak{B} \backslash \{\mathbf{0}\} : \beta \succsim_i \mathbf{0}\}$$

where the zero measure $\mathbf{0}$ represents the empty bundle. We know that $\mathfrak{B}_i \cup \{\mathbf{0}\}$ is closed by continuity of $\succsim_i$. Also, since a sequence of nonempty bundle with $\beta(R) \geq 1$ cannot converge to the empty bundle, we know that $\mathfrak{B}_i$ is closed in $\bar{\mathfrak{B}}$, and therefore compact.

9.3.1 Proof of Proposition 4.3

Now I prove the regularity of the matching game induced by the large-economy model with small contracts.

1. Compactness of the allocation space $\mathcal{M}$

Let

$$\mathcal{M}_i := \{\text{Borel measure } \mu_i \text{ on } \mathfrak{B}_i : \mu_i(\mathfrak{B}_i) \leq m_i\}$$

For an allocation $\mu \in \mathcal{M}$, we have $\mu_i \in \mathcal{M}_i$ by the total mass constraint, and therefore we have $\mathcal{M} \subset \bar{\mathcal{M}} := \prod_{i \in I} \mathcal{M}_i$.

Because $\mathfrak{B}_i$ is a compact metrizable spaces, we know that $\mathcal{M}_i$ with the weak-* topology is compact and metrizable. Then the product space $\bar{\mathcal{M}}$ endowed with the weak-* topology is also compact and metrizable. Therefore, to show $\mathcal{M}$ to be compact, it is sufficient to show that $\mathcal{M}$ is sequentially closed in $\bar{\mathcal{M}}$. Arbitrarily take a sequence $\left(\mu^k\right)$ in $\mathcal{M}$ convergent to $\mu^0 \in \bar{\mathcal{M}}$, I want to show (1) the total mass constraint $\mu_i^0\left(\mathfrak{B}_i\right) \leq m_i$ for each i, and (2) there exists Borel measure μ_x^0 over X s.t. the accounting identity $\sum_{i \in I} \int_{\beta \in \mathfrak{B}_i} \beta d\mu_i^0 = \int_{x \in X} x d\mu_x^0$ holds.

To show (1), notice that $\mu_i^k\left(\mathfrak{B}_i\right) = \int_{\mathfrak{B}_i} 1 d\mu_i^k \to \int_{\mathfrak{B}_i} 1 d\mu_i^0 = \mu_i^0\left(\mathfrak{B}_i\right)$ because the constant function 1 is continuous on $\mathfrak{B}_i$. Because $\mu_i^k\left(\mathfrak{B}_i\right) \leq m_i$ due to $\mu^k \in \mathcal{M}$, we have $\mu_i^0\left(\mathfrak{B}_i\right) \leq m_i$.

To show (2), let μ_x^k be the Borel measure over X that corresponds to each allocation μ^k. Define

$$\mathcal{M}_x := \left\{ \text{Borel measure } \mu_x \text{ on } X : \mu_x\left(X\right) \leq \frac{N \sum_{i \in I} m_i}{x_{\min}} \right\}$$

where $x_{\min} := \min_{x \in X} x\left(R\right)$ is well-defined and positive, because $x\left(R\right)$ is continuous in x, and X is a compact set that does not contain the zero measure. For each μ_x that corresponds to a feasible allocation, we have $\mu_x \in \mathcal{M}_x$ because

$$\begin{aligned} N \sum_{i \in I} m\left(i\right) &= \sum_{i \in I} \int_{\beta \in \mathfrak{B}_i} N d\mu_i \\ &\geq \sum_{i \in I} \int_{\beta \in \mathfrak{B}_i} \beta(R) d\mu_i = \int_{x \in X} x(R) d\mu_x \\ &\geq \int_{x \in X} x_{\min} d\mu_x = x_{\min} \mu_x\left(X\right) \end{aligned}$$

Because X is a compact metrizable spaces, we know that $\mathcal{M}_x$ with the weak-* topology is compact and metrizable. Therefore the sequence $\left(\mu_x^k\right)$ has a subsequence $\left(\mu_x^{k_l}\right)$ convergent to some $\mu_x^0 \in \mathcal{M}_x$. Now I claim that $\sum_{i \in I} \int_{\beta \in \mathfrak{B}_i} \beta d\mu_i^0 = \int_{x \in X} x d\mu_x^0$.

Take an arbitrary $f \in C\left(R\right)$. Because $\mu_i^k \to \mu_i^0$, we have

$$\int_{\beta \in \mathfrak{B}_i} \left(\int_R f d\beta \right) d\mu_i^k \to \int_{\beta \in \mathfrak{B}_i} \left(\int_R f d\beta \right) d\mu_i^0$$

since $\int_R f d\beta$ is a continuous function in β, which is in turn because f is a continuous function

on R. By accounting identity for each allocation μ^k, we have

$$\int_{x\in X}\left(\int_R f dx\right) d\mu_x^k = \sum_{i\in I}\int_{\beta\in\mathfrak{B}_i}\left(\int_R f d\beta\right) d\mu_i^k \to \sum_{i\in I}\int_{\beta\in\mathfrak{B}_i}\left(\int_R f d\beta\right) d\mu_i^0$$

Along the subsequence indexed by l, we have

$$\int_{x\in X}\left(\int_R f dx\right) d\mu_x^{k_l} \to \int_{x\in X}\left(\int_R f dx\right) d\mu_x^0$$

because $\int_R f dx$ is a continuous function in x, which is in turn because f is a continuous function on R. Since the limit of the subsequence has to be the same as the limit of the whole sequence, we have

$$\sum_{i\in I}\int_{\beta\in\mathfrak{B}_i}\left(\int_R f d\beta\right) d\mu_i^0 = \int_{x\in X}\left(\int_R f dx\right) d\mu_x^0$$

which is the accounting identity I want to show.

2. Continuity of the blocking relation $\sqsupset_i$

For each agent type i and allocation $\hat{\mu}\in\mathcal{M}_i$, by definition the set

$$\{\mu\in\mathcal{M}:\hat{\mu}\sqsupset_i\mu\} = \left\{\mu\in\mathcal{M}:\mu_i(\mathfrak{B}_i)=m_i \text{ and } \mu_i\left(\left\{\beta\in\mathfrak{B}_i:\underline{\beta}_i(\hat{\mu})\succ_i\beta\right\}\right)=0\right\}$$

Because $\mathcal{M}$ is metrizable, it is sufficient to show the set above to be sequentially closed in $\mathcal{M}$. Arbitrarily take a sequence $\left(\mu^k\right)$ of allocations in the set above that is convergent to a allocation $\mu^0\in\mathcal{M}$, I want to show that the limiting allocation μ^0 is also in the set, i.e. $\mu_i^0(\mathfrak{B}_i)=m_i$ and $\mu_i^0\left(\left\{\beta\in\mathfrak{B}_i:\underline{\beta}_i(\hat{\mu})\succ_i\beta\right\}\right)=0$. Because $\mu^k\to\mu^0$ implies $\mu_i^k\to\mu_i^0$, we have $\mu_i^k(\mathfrak{B}_i)\to\mu_i^0(\mathfrak{B}_i)$. Because $\mu_i^k(\mathfrak{B}_i)=m_i$ for each k, we have $\mu_i^0(\mathfrak{B}_i)=m_i$. On the other hand, continuity of $\succsim_i$ implies that $\left\{\beta\in\mathfrak{B}_i:\hat{\beta}_i\succ_i\beta\right\}$ is an open set in $\mathfrak{B}_i$. By Portmanteau theorem[14] of weak convergence, we have

$$\liminf\mu_i^k\left(\left\{\beta\in\mathfrak{B}_i:\underline{\beta}_i(\hat{\mu})\succ_i\beta\right\}\right)\geq\mu_i^0\left(\left\{\beta\in\mathfrak{B}_i:\underline{\beta}_i(\hat{\mu})\succ_i\beta\right\}\right)$$

Because $\mu_i^k\left(\left\{\beta\in\mathfrak{B}_i:\hat{\beta}\succ_i\beta\right\}\right)=0$ for each k, we have $\mu_i^0\left(\left\{\beta\in\mathfrak{B}_i:\hat{\beta}\succ_i\beta\right\}\right)=0$. This completes the proof of Proposition 4.3.

[14] See, for example, Ash (1972), Theorem 4.5.1.

9.4 Proof of Proposition 6.3

Let's check the three requirements of a convex matching game.

(1) The matching space $\mathcal{M}$ is a subset of the vector space $\left\{\mu = (\mu_f)_{f\in F} : \mu_f \in \mathbb{R}^{\Theta}\right\}$ with addition and scalar multiplication defined component-wise. Clearly, the empty matching μ^0 is the zero vector of the vector space.

(2) If $\sum_{j=1}^m w^j \phi(\mu^j) \leq \mathbf{1}$, where $w^j > 0$ and $\mu^j \in \mathcal{M}$ for all $j = 1, 2, \ldots, m$, then the linear combination $\mu := \sum_{j=1}^m w^j \mu^j$ is also a feasible IR matching. To see this, first we have

$$\sum_{f\in F} \mu_f(\theta) = \sum_{f\in F}\sum_{j=1}^m w^j \mu_f^j(\theta) = \sum_{j=1}^m w^j \left(\sum_{f\in F} \mu_f^j(\theta)\right) = m(\theta)\sum_{j=1}^m w^j \phi_\theta(\mu) \leq m(\theta)$$

for each θ, and therefore μ is a feasible matching. To see IR, for each firm f we have

$$\mu_f = \sum_{j=1}^m w^j \mu_f^j = \sum_{j:\phi_f(\mu^j)=1} w^j \mu_f^j \succsim_f 0$$

by convexity of $\succsim_f$ because $\sum_{j:\phi_f(\mu^j)=1} w^j = \sum_{j=1}^m w^j \phi_f(\mu^j) \leq 1$ and $\mu_f^j \succsim_f 0$ for each j with $\phi_f(\mu^j) = 1$.

On the other hand, for each worker type θ with $\mu_f(\theta) > 0$, I need to show $(f, \mu_f) \succsim_\theta \emptyset$. Because $\mu_f(\theta) > 0$, there exists $\hat{\jmath}$ s.t. $\mu_f^{\hat{\jmath}}(\theta) > 0$. Then by competition aversion, for every IR match (f, x) with $x(\theta) = 0$ we have $(f, x) \succsim_\theta (f, \mu_f^{\hat{\jmath}}) \succsim_\theta \emptyset$. Therefore, we have $(f, \mu_f) \succsim_\theta \emptyset$ by convexity of $\succsim_\theta$ because

$$\mu_f = \sum_{j:\phi_f(\mu^j)=1,\mu_f^j(\theta)>0} w^j \mu_f^j + \sum_{j:\phi_f(\mu^j)=1,\mu_f^j(\theta)=0} w^j \mu_f^j + \left(1 - \sum_{j:\phi_f(\mu^j)=1} w^j\right)\cdot 0$$

Therefore, the linear combination $\mu := \sum_{j=1}^m w^j \mu^j$ is a feasible IR matching.

(3) For each firm f, define the relation $\trianglerighteq_f$ over $\mathcal{M}_f$ s.t. $\mu' \trianglerighteq_f \mu$ if $\mu'_f \succsim_f \mu_f$. For each worker type θ, define the relation $\trianglerighteq_\theta$ over $\mathcal{M}_\theta$ s.t. $\mu' \trianglerighteq_\theta \mu$ if there exists f with $\mu_f(\theta) > 0$ s.t. $(f', \mu'_{f'}) \succsim_\theta (f, \mu_f)$ for all f' with $\mu'_{f'}(\theta) > 0$. In other words, the relation $\trianglerighteq_\theta$ is obtained by comparing the worst position for type θ workers under two matchings.

For an arbitrary $i \in F \cup \Theta$, let's consider a ϕ_i-convex combination $\mu := \sum_{j=1}^m w^j \mu^j$, i.e. $w^j > 0$ and $\mu^j \in \mathcal{M}$ for all j s.t. $\sum_{j=1}^m w^j \phi_i(\mu^j) = 1$ and $\sum_{j=1}^m w^j \phi(\mu^j) \leq \mathbf{1}$. Further assume that $\mu^j \trianglerighteq_i \hat{\mu}$ for each of its component $\mu^j \in \mathcal{M}_i$. Clearly, μ is a feasible IR matching by (2), and it is sufficient to show that $\hat{\mu} \not\trianglerighteq_i \mu$.

If $i \in F$, this is trivial because

$$\mu_f = \sum_{j:\phi_f(\mu^j)=1} w^j \mu_f^j \succsim_f \hat{\mu}$$

by convexity of $\succsim_f$, because $\sum_{j:\phi_f(\mu^j)=1} w^j = \sum_{j=1}^m w^j \phi_i(\mu^j) = 1$ and $\mu_f^j \succsim_f \hat{\mu}$ for each j with $\phi_f(\mu^j) = 1$.

If $i \in \Theta$, by definition we need to show $\sum_{f \in F} \mu_f(\theta) = m(\theta)$ and there exists $\hat{f}$ with $\hat{\mu}_{\hat{f}}(\theta) > 0$ s.t. $(f, \mu_f) \succsim_\theta (\hat{f}, \hat{\mu}_{\hat{f}})$ for all $f \in F$ with $\mu_f(\theta) > 0$. First, we have

$$\sum_{f \in F} \mu_f(\theta) = \sum_{f \in F} \sum_{j=1}^m w^j \mu_f^j(\theta) = \sum_{j=1}^m w^j \left(\sum_{f \in F} \mu_f^j(\theta) \right) = m(\theta) \sum_{j=1}^m w^j \phi_\theta(\mu) = m(\theta)$$

Second, let $(\hat{f}, \hat{\mu}_{\hat{f}})$ be the worst position for type θ students in matching $\hat{\mu}$, i.e.

$$(\hat{f}, \hat{\mu}_{\hat{f}}) \in \arg\min_{\succsim_\theta} \{(f, \hat{\mu}_f) : \hat{\mu}_f(\theta) > 0\}$$

Then for each f with $\mu_f(\theta) > 0$, there exists $\hat{\jmath}$ s.t. $\mu_f^{\hat{\jmath}}(\theta) > 0$. By competition aversion, for every IR match (f, x) with $x(\theta) = 0$ we have $(f, x) \succsim_\theta (f, \mu_f^{\hat{\jmath}}) \succsim_\theta \emptyset$. Therefore we have $(f, \mu_f) \succsim_\theta (\hat{f}, \hat{\mu}_{\hat{f}})$ by convexity of $\succsim_\theta$, because

$$\mu_f = \sum_{j:\phi_f(\mu^j)=1, \mu_f^j(\theta)>0} w^j \mu_f^j + \sum_{j:\phi_f(\mu^j)=1, \mu_f^j(\theta)=0} w^j \mu_f^j + \left(1 - \sum_{j:\phi_f(\mu^j)=1} w^j \right) \cdot 0$$

9.5 Scarf's Lemma Approach to Stability

In this appendix, I demonstrate how we may show the existence of stable matchings by showing nonempty core under some modified preferences. The model I study in this appendix is a two-sided many-to-one matching model with a continuum of firms and workers, which is a special case of the first model in Azevedo and Hatfield (2015).

Consider a continuum of firms and a continuum of workers. Let F be the finite set of firm types, and Θ be the finite set of worker types. Let $m(f) > 0$ be the mass of type f firms, and $m(\theta) > 0$ be the mass of type θ workers. Each coalition consists of one firm and finitely many workers, and a coalition type is denoted as x, which is an nonnegative integer vector over $F \cup \Theta$. We require that $x(f) = 1$ if a type x coalition involves a type f firm, and 0 otherwise. For each worker type θ, the integer $x(\theta)$ is the number of type θ workers involved in a type x coalition. Assume that there are only finitely many types of coalitions

that are acceptable to all their members, and let X be the set of all such coalition types.

An allocation μ is a nonnegative mass vector over X, and $\mu(x)$ represents the mass of type x contracts present under the allocation μ. Feasibility requires the total mass constraint $\sum_{x\in X}\mu(x)\cdot x(f)\leq m(f)$ for each firm type f and $\sum_{x\in X}\mu(x)\cdot x(\theta)\leq m(\theta)$ for each worker type θ. Let X_f be the set of coalition types that involve a type f firm, and preferences $\succsim_f$ of type f firms are defined over X_f, i.e. firms value the composition of their workers. Let $\succsim_\theta$ be the preferences of type θ workers over firms, i.e. the workers only value the firm that hires them. We extend $\succsim_\theta$ to coalition types according to the firm type they contain.

The following notion of core is standard.

Definition 9.4 *A coalition type $\hat{x}$* ***c-blocks*** *an allocation μ, if*

(1) For the firm type f involved in $\hat{x}$, either not all type-f firms are matched, or there exists x that also involves f s.t. $\mu(x)>0$ and $\hat{x}\succ_f x$, and

(2) For each worker type θ with $\hat{x}(\theta)>0$, either not all type-θ workers are matched, or there exists x with $\mu(x)>0$ and $x(\theta)>0$ s.t. $\hat{x}\succ_\theta x$.

An allocation is in the ***core*** *if it is not c-blocked by any coalition type.*

We can also define the following notion of stability.

Definition 9.5 *A coalition type $\hat{x}$* ***s-blocks*** *an allocation μ, if for the firm type f involved in $\hat{x}$, either*

(1) There exists x that also involves f s.t. $\mu(x)>0$ and $\hat{x}\succ_f x$, and for each worker type θ with $\hat{x}(\theta)>x(\theta)$, there exist type-$\theta$ workers who are either unmatched or matched to a strictly less preferred firm under μ, or

(2) There exists unmatched type-f firms, and for each worker type θ with $\hat{x}(\theta)>0$, there exist type-θ workers who are either unmatched or matched to a strictly less preferred firm under μ.

In the blocking notion of stability, a firm may bring some of its existing workers into a blocking coalition, although these workers are indifferent. In the blocking notion of the core, however, all participants of a blocking coalition are required to strictly benefit from the block. To connect these two solution concepts, let's consider a modification of workers' preferences by using the relevant firm's preferences to break ties. Formally, the modified workers' preference relation $\succsim'_\theta$ is the same as the original $\succsim_\theta$ when comparing two coalition types that involve different types of firms, but follows $\succsim_f$ when comparing two coalition types that both involve f.

Then the following observation is straightforward.

Proposition 9.6 *If allocation μ is in the core under the modified preferences $\succsim'$, then μ is stable under the original preferences $\succsim$.*

Proof. Suppose that there exists $\hat{x}$ that s-blocks allocation μ under the original preferences $\succsim$. I want to show that $\hat{x}$ also c-blocks μ under the modified preferences $\succsim'$, which would contradict the assumption in the proposition and conclude the proof.

First, we know that $\succ'_f = \succ_f$, and so the firm wants to participate in the block. The new workers in the s-block is willing to participate in the block by definition. The existing workers strictly prefer the new coalition under the modified preferences, because they follow the preferences of the firm. Therefore $\hat{x}$ c-blocks allocation μ under $\succsim'$. Contradiction. ■

By Theorem 4.4, the core under the modified preferences is nonempty. Therefore, stable matchings exist under the original preferences, and this result, stated below, is a special case of Azevedo and Hatfield (2015).

Theorem 9.7 (Azevedo and Hatfield, 2015) *In the two-sided many-to-one matching model with a continuum of firms and workers, stable matchings exist.*

9.6 Generalized Cooperative Games

In this appendix, I explore the maximum scope of the Scarf's lemma approach, and provide a more general structure to which Scarf's lemma can be applied. This structure more general than convexity is a balancedness structure, which is a generalization of the pivotal balancedness condition introduced by Iehle (2007). I show that the generalized pivotal balancedness structure is necessary and sufficient for nonempty core.

Consider a generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i \in I}\right\}$. Set I is a finite set of players. In applications, each player $i \in I$ can represent either one agent or a continuum of identical agents. Set $\mathcal{M}$ represents the nonempty set of feasible and individually rational allocations, and set B is the set of potential blocks. For each potential block $b \in B$, its *participation set* $\mathcal{I}(b)$ with $\emptyset \neq \mathcal{I}(b) \subset I$ captures the set of players who need to agree on the block b for the block to be viable. Each player $i \in I$ is associated with a complete and transitive preference relation $\trianglerighteq_i$ on the set

$$B_i := \{b \in B : i \in \mathcal{I}(b)\}$$

i.e. the set of blocks that involve player i. Let the relation $\triangleright_i$ be the strict version of $\trianglerighteq_i$, and extend $\triangleright_i$ to $B_i \times \mathcal{M}$ subject to transitivity in the sense that $b' \trianglerighteq_i b$ and $b \triangleright_i \mu$ imply $b' \triangleright_i \mu$. Let's interpret the relation $\triangleright_i$ as a blocking relation for player i, whose meaning is elaborated in the definition below.

Definition 9.8 *In a generalized cooperative game* $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$*, an allocation* μ *is* ***blocked*** *by a block* b*, if* $b \triangleright_i \mu$ *for every* $i \in \mathcal{I}(b)$*, i.e. all participants of block* b *are willing participate under allocation* μ*.*

A allocation $\mu \in \mathcal{M}$ *is in the* ***core*** *if it is not blocked by any* $b \in B$*.*

Now I'm going to introduce for generalized cooperative games the following pivotal balancedness structure, which is a generalization of the pivotal balancedness condition in Iehle (2007) for coalition formation games with finitely many agents.

Definition 9.9 *A function* $\phi : B \to \mathbb{R}^I_+ / \{\mathbf{0}\}$ *is called a* ***pivotality function****.*

A generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ *is* ***pivotally balanced w.r.t. the pivotality function*** ϕ*, if for every finite set of blocks* $(b^j)_{j=1}^m$ *and positive weights* $(w^j)_{j=1}^m$ *s.t.* $\sum_{j=1}^m w^j \cdot \phi(b^j) \leq \mathbf{1}$*, there exists an allocation* $\mu^* \in \mathcal{M}$ *s.t. for each* $i \in I$ *with* $\sum_{j=1}^m w^j \cdot \phi_i(b^j) = 1$*, we have* $b^{\hat{\jmath}} \not\triangleright_i \mu^*$ *for some* $\hat{\jmath}$ *with* $i \in \mathcal{I}(b^{\hat{\jmath}})$*.*

We also say that a generalized cooperative game $\mathcal{G}$ is *pivotally balanced*, without specifying the pivotality function, if it is so w.r.t. some pivotality function. Now let's provide some intuition behind the terminology "pivotality". Suppose that an allocation μ^* is in the core, then define the pivotality function ϕ^{μ^*} induced by μ^* as

$$\phi_i^{\mu^*}(b) := \begin{cases} 1 \text{ if } i \in \mathcal{I}(b) \text{ and } b \not\triangleright_i \mu^* \\ 0, \text{ otherwise} \end{cases}$$

In the construction above, the binary vector $\phi^{\mu^*}(b)$ indicates the set of "pivotal" participants of block b, in the sense that these agents are pivotal to the success of the block because they are not willing to participate b under allocation μ^*. Because μ^* is in the core, we know that such participant exists, i.e. $\phi^{\mu^*}(b) \neq \mathbf{0}$, and so ϕ^{μ^*} is a valid pivotality function. With the construction of ϕ^{μ^*} above, let's state the necessity of pivotal balancedness for nonempty core.

Theorem 9.10 (Necessity) *If the generalized cooperative game* $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ *has an allocation* μ^* *in it core, then the game is pivotally balanced w.r.t. the pivotality function* ϕ^{μ^*}*.*

Proof. Arbitrarily take a finite set of blocks $(b^j)_{j=1}^n$ and positive weights $(w^j)_{j=1}^n$ s.t. $\sum_{j=1}^m w^j \cdot \phi^{\mu^*}(b^j) \leq \mathbf{1}$. I claim that the core allocation μ^* is the allocation we want to find. To see this, for each $i \in I$ with $\sum_{j=1}^m w^j \cdot \phi_i^{\mu^*}(b^j) = 1$, we know that there exists some j s.t. $\phi_i^{\mu^*}(b^j) = 1$, otherwise the equality cannot hold. This implies $b^j \not\triangleright_i \mu^*$ by the construction of ϕ^{μ^*}. ■

Now let's turn to the sufficiency of pivotal balancedness, and first make the following observation.

Lemma 9.11 *In a generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ is pivotally balanced, for each finite family $(b^j)_{j=1}^m$ of potential blocks, we can find an allocation $\mu \in \mathcal{M}$ that is not blocked by any b^j for $j = 1, 2, \ldots, m$.*

Proof. Let ϕ be the pivotality function for this game, and consider the $|I| \times m$ matrix $A := [\phi(b^1), \phi(b^2), \ldots, \phi(b^m)]$. Because $\phi(b) \neq \mathbf{0}$, each column of A has at least one positive entry, which satisfied the requirement of Scarf's lemma. For each row $i \in I$, let the domain of $\geq_i$ be $D_i := \{j : i \in \mathcal{I}(b^j)\}$, and let $j \geq_i j'$ if $b^j \trianglerighteq_i b^{j'}$. Invoking Scarf's lemma for the matrix A, we obtain a dominating weight vector $\hat{w}$. Then we invoke the pivotal balancedness condition for $(b^j)_{j=1}^m$ and $(\hat{w}^j)_{j=1}^m$, ignoring the weights that are zero if there is any, to obtain an allocation μ^*. ■

Now I claim that μ^* cannot be blocked by b^j for all $j = 1, 2, \ldots, m$. To see this, let i be the row at which $\hat{w}$ dominates column j. By definition of domination, we know that $i \in \mathcal{I}(b^j)$, $\sum_{k=1}^m \hat{w}^k \cdot \phi_i(b^j) = 1$, and $b^k \trianglerighteq_i b^j$ for all k with $i \in \mathcal{I}\left(b^k\right)$ and $\hat{w}_k > 0$. Then by pivotal balancedness, $\sum_{k=1}^m \hat{w}^k \cdot \phi_i(b^j) = 1$ implies $b^k \not\triangleright_i \mu^*$ for some k with $i \in \mathcal{I}\left(b^k\right)$ and $\hat{w}^k > 0$. Therefore we have $b^j \not\triangleright_i \mu^*$ by transitivity of $\triangleright_i$, and so the allocation μ^* is not blocked by b^j.

As summary of the proof above, the first step is to construct the matrix A whose columns are the pivotality vectors of the blocks. Then the second step is to use Scarf's lemma to obtain a dominating weight vector $\hat{w}$. The third step is to invoke pivotal balancedness to convert the dominating vector $\hat{w}$ to the unblocked allocation μ^*.

In many applications, the set B of potential blocks we need to rule out is usually infinite. In this case, we need the following regularity condition to go from finitely many blocks to infinitely many blocks.

Definition 9.12 *A generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ is **regular** if the allocation space $\mathcal{M}$ is a compact topological space, and the blocking relation $\triangleright_i$ is upper semi-continuous for each $i \in I$ in the sense that the upper contour set $\{\mu \in \mathcal{M} : b \not\triangleright_i \mu\}$ is closed in $\mathcal{M}$ for all $b \in B_i$.*

Then we have the following sufficiency result.

Theorem 9.13 (Sufficiency) *If a generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ is regular and pivotally balanced, then its core is nonempty.*

Proof. Let the core be denoted as $\mathcal{M}^*$, and let $\mathcal{M}_b^*$ be the set of allocations that cannot be blocked by b. By definition, the core can be represented

$$\mathcal{M}^* = \bigcap_{b \in B} \mathcal{M}_b^*$$

Because the allocation space $\mathcal{M}$ is compact by regularity, and the intersection of any finitely many $\mathcal{M}_b^*$'s is nonempty by Lemma 9.11, to show nonemptiness of $\mathcal{M}^*$, it is sufficient to show that $\mathcal{M}_b^*$ is closed in $\mathcal{M}$ for each potential block $b \in B$. The closedness of $\mathcal{M}_b^*$ is a result of upper semi-continuity of the blocking relations. To see this, notice that

$$\mathcal{M}_b^* = \bigcup_{i \in \mathcal{I}(b)} \{\mu \in \mathcal{M} : b \not\rhd_i \mu\}$$

By upper semi-continuity of $\rhd_i$, the set $\{\mu \in \mathcal{M} : b \not\rhd_i \mu\}$ is closed in $\mathcal{M}$, and so the finite union is also closed in $\mathcal{M}$. ■

Therefore, we have shown that under regularity, the pivotal balancedness condition is necessary and sufficient for nonempty core in a generalized cooperative game.

The pivotal balancedness structure introduced in this appendix is more general than the convexity structure introduced in the paper. To see this, a convex matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$ induces a generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i \in I}\right\}$, where $B := \mathcal{M} \setminus \{\mu^0\}, \mathcal{I}(\mu) := \{i \in I : \phi_i(\mu) > 0\}$, the relation $\trianglerighteq_i$ is the one whose existence is guaranteed by Definition 3.2(3), and the relation $\rhd_i$ on from B_i to $\mathcal{M}$ is equal to $\sqsupset_i$. By comparing definitions, it is not difficult to see that the generalized cooperative game induced by a convex matching game $G = \left\{I, \mathcal{M}, \phi, (\sqsupset_i)_{i \in I}\right\}$ is pivotally balanced w.r.t. G's characteristic function ϕ. In addition, if the matching game is regular, then the induced generalized cooperative game is also regular. In this sense, the sufficiency result 9.13 is more general than Theorem 3.4.

However, the pivotal balancedness structure is less intuitive than the convexity structure, and it is more difficult to check in applications. To check pivotal balancedness, we need to examine all pivotality functions $\phi : B \to \mathbb{R}_+^I / \{\mathbf{0}\}$, which is difficult, if not impossible.

9.6.1 Relation to Scarf's Balancedness Condition

In this subsection, I explain the connection between the pivotal balancedness condition and Scarf's balancedness condition. As we are going to see, the Scarf's balancedness condition corresponds to the pivotal balancedness condition w.r.t. a particular pivotality function. This also partly explains why Scarf's balancedness condition is sufficient but not necessary

for nonempty core in an NTU game, as is shown by Bogomolnaia and Jackson (2002).

Let's recall the setup of the finite NTU game studied by Scarf (1967). There is a finite set I of agents, and a coalition is a nonempty subset $S \subset I$, and $V_S \subset \mathbb{R}^S$ represent the set of payoff vectors that can be achieved by coalition S. It is assumed that (1) V_S is a closed set in $\mathbb{R}^S$, (2) $u \in V_S$ and $u' \leq u$ implies $u' \in V_S$, and (3) the set $\{u \in V_S : u_i \geq \underline{u}_i$ for all $i \in S\}$ is nonempty and bounded, where $\underline{u}_i$ is the maximum utility agent i can get by staying alone.

Definition 9.14 *A payoff vector $u \in V_I$ is **blocked** by coalition S, if there exists $\hat{u} \in V_S$ s.t. $\hat{u}_i > u_i$ for all $i \in S$. A payoff vector $u \in V_I$ is in the **core** if it is not blocked by any coalition.*

In the definition of core above, notice that individual rationality is handled by considering blocking by the singleton coalition $\{i\}$.

We say that a family of coalitions $(S^j)_{j=1}^m$ is *balanced* if there exist nonnegative weights $(w^j)_{j=1}^m$ s.t. $\sum_{j:i\in S^j} w^j = 1$ for each $i \in I$. Scarf (1967) provides the following balancedness condition that is sufficient but not necessary for nonempty core.

Definition 9.15 (Scarf's Balancedness Condition) *A finite NTU game is **balanced**, if for every balanced family $(S^j)_{j=1}^m$ of coalitions, the payoff vector $u \in \mathbb{R}^I$ must be in V_I whenever $u_{S^j} \in V_{S^j}$ for each $j = 1, 2, \ldots, m$, where u_{S^j} is the projection of $u \in \mathbb{R}^I$ to $\mathbb{R}^{S^j}$.*

The main theorem of Scarf (1967) is stated below.

Theorem 9.16 (Scarf, 1967) *A Scarf-balanced finite NTU game always has a nonempty core.*

To see its connection to the abstract framework, a finite NTU game induces a generalized cooperative game $\mathcal{G} = \left\{I, \mathcal{M}, B, \mathcal{I}, (\trianglerighteq_i)_{i\in I}\right\}$ as follows. Let the set I of players simply be the set of agents, and define the allocation space $\mathcal{M} := \{\, u \in V_I : u \geq \underline{v}\}$. The set of potential blocks is

$$B := \{(S, \hat{u}) : \emptyset \neq S \subset I \text{ and } \hat{u} \in V_S\}$$

For each potential block $(S, \hat{u})$, its participation set $\mathcal{I}\left((S, \hat{u})\right)$ is simply the coalition S. For each agent $i \in I$, the relation $\trianglerighteq_i$ is defined over $B_i := \{(S, \hat{u}) : i \in S \subset I \text{ and } \hat{u} \in V_S\}$ s.t. $(S', \hat{u}') \trianglerighteq_i (S, \hat{u})$ iff $\hat{u}'_i \geq \hat{u}_i$. Extend the corresponding strict relation $\triangleright_i$ to $B_i \times \mathcal{M}$ s.t. $(S, \hat{u}) \triangleright_i u$ iff $\hat{u}_i \geq u_i$.

In the generalized cooperative game $\mathcal{G}$ induced by Scarf's finite NTU game, it is straightforward to verify that the core defined in Definition 9.8 reduces to that in Definition 9.14.

The next result establishes the connection between Scarf's balancedness condition and pivotal balancedness condition.

Proposition 9.17 *The finite NTU game satisfies the Scarf's balancedness condition iff the generalized cooperative game induced by the finite NTU game is pivotally balanced w.r.t. the pivotality function $\hat{\phi}$ defined as*[15]

$$\hat{\phi}_i\left((S,\hat{u})\right) = \begin{cases} 1, \text{ if } i \in S \\ 0, \text{ otherwise} \end{cases}$$

Proof. "If" part:

Arbitrarily take a balanced family $(S^j)_{j=1}^m$ of coalitions, and $u \in \mathbb{R}^I$ s.t. $u_{S^j} \in V_{S^j}$ for each j. I want to show that $u \in V_I$. Let $(w^j)_{j=1}^m$ be a set of nonnegative weights s.t. $\sum_{j:i\in S^j} w^j = 1$ for each $i \in I$. Rewriting these equalities in terms of the pivotality function, we have $\sum_{j=1}^m w^j \cdot \hat{\phi}\left(S^j, u_{S^j}\right) = \mathbf{1}$. Invoking the pivotal balancedness condition for $(S^j, u_{S^j})_{j=1}^m$ and $(w^j)_{j=1}^m$ while ignoring the weights that are zero if there is any, we obtain an allocation $u^* \in V_I$ s.t. $u_i \leq u_i^*$ for all i. Then we know that $u \in V_I$.

"Only if" part:

Arbitrarily take a finite set of blocks $(S^j, \hat{u}^j)_{j=1}^m$ and positive weights $(w^j)_{j=1}^m$ s.t. $\sum_{j=1}^m w^j \cdot \hat{\phi}_i\left((S^j,\hat{u}^j)\right) \leq \mathbf{1}$, I want to find an allocation $u^* \in V_I$ s.t. for each $i \in I$ with $\sum_{j=1}^m w^j \cdot \hat{\phi}_i\left((S^j,\hat{u}^j)\right) = 1$, we have $\hat{u}_i^{\hat{j}} \leq u_i^*$ for some $\hat{j}$ with $i \in S^{\hat{j}}$. By construction of $\hat{\phi}$, the inequalities reduce to $\sum_{j:i\in S^j} w^j \leq 1$ for all $i \in I$. Although the family $(S^j)_{j=1}^m$ of coalitions may not be balanced, it becomes balanced if we add the singleton coalitions $\{i\}$ for agent i's with $\sum_{j:i\in S^j} w^j < 1$, and set the weight to be $1 - \sum_{j:i\in S^j} w^j$.

Define the payoff vector $u^* \in \mathbb{R}^I$ as

$$u_i^* := \begin{cases} \min\left\{\hat{u}_i^j : S^j \ni i\right\}, \text{ if } \sum_{j:i\in S^j} w^j = 1 \\ \min\left\{\hat{u}_i^j : S^j \ni i\right\} \cup \{\underline{u}_i\}, \text{ if } \sum_{j:i\in S^j} w^j < 1 \end{cases}$$

By construction we have $u_{S^j}^* \in V_{S^j}$ for all j, and $u_i^* \in V_{\{i\}}$ for all i with $\sum_{j:i\in S^j} w^j < 1$. By Scarf's balancedness condition, we know that $u^* \in V_I$. Also, for each $i \in I$ with $\sum_{j=1}^m w^j \cdot \hat{\phi}_i\left((S^j,\hat{u}^j)\right) = \sum_{j:i\in S^j} w^j = 1$, we have $u_i^* = \min\left\{\hat{u}_i^j : S^j \ni i\right\} \geq \hat{u}_i^{\hat{j}}$ for some $\hat{j}$ with $i \in S^{\hat{j}}$. ■

Proposition 9.17 above states that Scarf's balancedness condition is equivalent to pivotal balancedness w.r.t. a particular pivotality function $\hat{\phi}$. As a consequence, Scarf's balancedness condition is stronger than pivotal balancedness in general. This partly explains why

[15] Notice that the function $\hat{\phi}$ is indeed a valid pivotality function, because $\phi\left((S,\hat{u})\right) \neq \mathbf{0}$ due to the fact that the coalition S is nonempty.

Scarf's balancedness is sufficient but not necessary for nonempty core in a general NTU game.

Also, Proposition 9.17 implies that the generalized cooperative game $\mathcal{G}$ induced by a Scarf-balanced finite NTU game is pivotally balanced. Therefore, to replicate the main theorem of Scarf (1967), it is sufficient to verify that $\mathcal{G}$ is also regular. To see this, the allocation space $\mathcal{M} := \{ u \in V_I : u \geq \underline{v}\}$ is closed and bounded in $\mathbb{R}^I$ w.r.t. Euclidean metric, and therefore compact. On the other hand, for each block $(S, \hat{u})$ and agent $i \in S$, the upper contour set

$$\{u \in \mathcal{M} : (S, \hat{u}) \not\succ_i u\} = \{ u \in \mathcal{M} : u_i \geq \hat{u}_i\}$$

which is closed in $\mathcal{M}$.

Therefore, with pivotal balancedness and regularity, we replicate the main theorem of Scarf (1967) using Theorem 9.13.

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