Core of Convex Matching Games: A Scarf's Lemma Approach*

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Abstract

As a central solution concept of cooperative games, the notion of the core is widely studied and applied in the matching theory literature. A matching outcome is said to be in the core if no coalition of agents can find a profitable joint deviation. However, it is well known that the core may be empty with general contracting networks, multi-lateral contracts, or complementary preferences. Fortunately, recent studies including Hatfield and Kominers (2015), Azevedo and Hatfield (2018), and Che, Kim, and Kojima (2019) obtain nonempty core results under different assumptions despite those difficulties. In this paper, we identify a convexity structure of matching games that unifies our understanding of those nonempty core results and highlights their relation to a lemma of Scarf (1967). This approach also allows us to obtain a new nonempty core result after introducing peer preferences into the model of Che, Kim, and Kojima (2019).

KEYWORDS: Core, convexity, contracting networks, Scarf's lemma.

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1 Introduction

The solution concept of the *core* and its close relative, *stability*, have been widely studied and applied in two-sided matching problems such as marriage and school choice. We say that a matching outcome, or simply a *matching*, is *blocked* by a coalition of agents if those agents can find a profitable joint deviation. In the context of the marriage problem (or the school choice problem), a coalition consists of a woman and a man (or a school and a student), who block the matching in question if they are not matched to each other but they can be made strictly better off by doing so. A matching is the *core* if it is not blocked by any coalition. If we are willing to assume that agents can perfectly coordinate to carry out profitable joint deviations, we may reasonably expect matchings outside the core to be less likely to occur than those in the core.

Like all other solution concepts, the notion of the core loses its predictive power if the core is empty. Although Gale and Shapley (1962) show that the core is always nonempty in the marriage problem, it may be empty in matching problems without a two-sided market structure. To understand the empty core issue, let us consider the following roommate problem adapted from Gale and Shapley (1962).

Example 1 (Three-individual roommate example)**.** Consider three individuals, A, B, and C. Any two of them can become roommates, in which case the third person will have to live alone. All of them prefer having a roommate to living alone, but A likes living with B better than C. Similarly, B prefers C to A, and C prefers A to B. Note that the matching in which A and B live together and C lives alone is blocked by individuals B and C since they will be strictly better off by becoming roommates. By symmetry, we also see that the matching in which B and C live together is blocked by the coalition formed by C and A, and the matching in which C and A live together is blocked by the coalition formed by A and B. Therefore, the core is empty since we have examined all three possible matchings and found that none of them is in the core.

It is also known that even the two-sided structure of a market does not always guarantee a nonempty core if matching is many-to-one. In matching problems such as labor markets or school choice in which each agent on one side of the market (firms or schools) is matched to multiple agents on the other side (workers or students), the core is guaranteed to be nonempty only under substitutable preferences (Kelso and Crawford (1982), Roth (1984)). In other words, if some firm (or school) regards some workers (or students) as complements, the core may again be empty. To illustrate, let us consider a labor market example taken from Che, Kim, and Kojima (2019), which has the similar nonempty core issue to Example 1.

Example 2 (Two-by-two labor market example)**.** Consider a labor market with two firms and two workers. Firm I regards the two workers as complements and would rather hire neither of them if it cannot hire both. Firm II wants to hire only one worker and prefers worker 1 to worker 2. Worker 1 prefers working for firm I, and worker 2 prefers working for firm II. Which matching is in the core in this example? First, firm I hiring both workers is not in the core since it is blocked by firm II and worker 2. Second, firm I hiring no one and firm II hiring worker 2 is not in the core either since it is blocked by firm II and worker 1. The last candidate is firm I hiring no one and firm II hiring worker 1, which is still not in the core since it is blocked by the coalition formed by firm I, worker 1, and worker 2. Therefore, the core is empty since we have examined all three possible matchings and found that none of them is in the core.

In the general equilibrium literature, it is known that the empty core issue tends to vanish when the market contains many consumers despite the difficulty of nonconvex preferences, which we know usually leads to en empty core in finite markets (Shapley and Shubik (1966), Aumann(1966)). Generalizing the intuition behind this result, we may conjecture that it is also easier to obtain a nonempty core in large matching markets. Fortunately, this conjecture is indeed true for the roommate problem. Let us consider the following the continuum variant of Example 1 and refer to it as the continuum roommate problem hereafter.

Example 3 (Continuum roommate problem)**.** Consider a continuum version of the aforementioned roommate problem with three types of individuals instead of only three individuals. Each type of individuals is of mass 1 and has the same preferences as in the three-individual roommate example (Example 1). In addition, each individual considers matching with another individual of the same type to be unacceptable. This model can be viewed as an infinite replica of the three-individual roommate problem. Note that the matching in which all type A individuals are matched with type B individuals and all type C individuals are unmatched is not in the core for the same reason as in Example 1, i.e., type B and type C individuals can block this matching. A similar argument holds for the matching in which all type B individuals are matched with type C individuals and the matching in which all type C individuals are matched with type A individuals. However, what is different in this continuum roommate problem is that there are many more possible matchings to consider than those three above. Particularly, let us consider the $(1/2, 1/2, 1/2)$ combination of the three matchings above, i.e., the matching in which half of the type A individuals are matched with half of the type B individuals, half of the type B individuals are matched with half of the type C individuals, and half of the type C individuals are matched with half of the type A individuals. In fact, this matching is in the core. To verify this, note that a type A individual matched with a type B individual does not wish to participate in a block since she already has her top choice. On the other hand, although a type A individual matched with a type C individual does wish to form a block with some type B individual, no type B individual has an incentive to join the block because any type B individual is already matched with some other type A individual or some type C individual, who is an even more preferred roommate. After ruling out all other blocks using similar arguments, we conclude that this matching is indeed in the core.

In this example above, we essentially assume that each type of individuals is perfectly divisible by letting it be a continuum. This divisible nature allows us to consider weighted combinations of a few matchings, and it turns out that one such combination is in the core.[1] Given this observation, it might be tempting to think that the core is always nonempty in a market with a continuum of agents. However, this is not true. Let us provide a counterexample, which we will refer to as the large coalition formation problem hereafter.

Example 4 (Large coalition formation)**.** There are again three types of individuals, A, B, and C, each of which is of mass 1. Individuals wish to form coalitions of positive mass no greater than 2. When a type A individual is in a coalition (x_A, x_B, x_C), where $x_i \geq 0$ is the mass of type i individuals in the coalition, her utility is $(100 + 2\sqrt{x_B + 1} + \sqrt{x_C + 1}) \cdot \sqrt{x_A + 1}$ if $x_A + x_B + x_C \leq 2$ and is -1 if $x_A + x_B + x_C > 2$. The utility functions of the other two types of individuals are cyclic symmetric to type A's. That is, if $x_A + x_B + x_C \leq 2$, a type B individual's utility is $(100 + 2\sqrt{x_C + 1} + \sqrt{x_A + 1}) \cdot \sqrt{x_B + 1}$ and a type C individual's utility is $(100 + 2\sqrt{x_A + 1} + \sqrt{x_B + 1}) \cdot \sqrt{x_C + 1}$; otherwise, their utility is -1. Although the individuals' preferences are convex due to their concave utility functions, the core is in fact empty. To see this, by examining the marginal utility, we first note that it is always beneficial for all individuals of the same type to join one coalition

[1] To guarantee a nonempty core in roommate problems, it is in fact sufficient to have an even number of individuals of each type, which is shown by Tan (1991) and Aharoni and Fleiner (2003). An analogous result for roommate problems with transfers is shown by Chiappori, Galichon, and Salanié (2014). Going beyond roommate problems, we need a continuum of agents of each type because the weights that give us the matching in the core are not necessarily half integrals, i.e., $k/2$ for some integer k.

at any cost. Then, this continuum market is essentially equivalent to the three-individual roommate example (Example 1), which we know has an empty core.

From this example above, we understand that a continuum market is not sufficient for the applicability of our approach to a nonempty core. Although weighted combinations of matchings could be defined, we cannot find a matching in the core. On the other hand, our next example shows that a market with a continuum of agents is also not necessary for the applicability of our approach. We will refer to it as the time-share roommate problem hereafter.

Example 5 (Time-share roommate problem)**.** Let us reconsider Example 1, but now we allow for time-share matching, i.e., two individuals can spend some fraction of their time together. Consider the matching in which individual A spends half of her time with individual B, individual B spends half of her time with individual C, and individual C spends half of her time with individual A, and this matching is in fact in the core. To verify this, it is sufficient to note that although A wishes to spend more time with B, B would not be willing to do so at the cost of decreasing her time spent with C, who is her more preferred roommate. Similar to Example 3, this matching in the core can be viewed as a $(1/2, 1/2, 1/2)$ combination of the three matchings we considered in Example 1, which are not in the core.

Note that in the example above, each pair of individuals chooses their time spent together from the interval $[0, 1]$. This continuous structure allows us to consider weighted combinations of a few matchings and it turns out that one such combination is in the core, just as in the continuum roommate problem (Example 3). However, note that the interpretation of "combinations" here is different. In Example 3, a combination is taken over population size, while in Example 5, the combination is taken over time spent together by each pair of individuals.[2]

Given the similarity between Example 3 and 5, we may wonder whether we have a unified understanding of this approach that obtains a matching in the core by combining a few matchings not necessarily in the core. Before we turn to that, let us consider one more example from Che, Kim, and Kojima (2019), which we will refer to as the large-firm labor market problem hereafter.

[2]It might be tempting to reinterpret the matching we found in the core as a random matching among three individuals, in which each pair is matched with probability $1/2$. However, this interpretation is not valid because individuals A and B being matched with probability $1/2$ implies that individual C is unmatched with probability at least $1/2$. However, this probability-share interpretation is valid in two-sided matching markets thanks to the Birkhoff-von Neumann theorem (see, for example, Hylland and Zeckhauser (1979), Bogomolnaia and Moulin (2001), Budish, Che, Kojima, and Milgrom (2013)).

Example 6 (Large-firm labor market)**.** Consider a variant of Example 2. There are still two firms, but there are two types of workers, each of which is of mass 1. This captures the asymmetry of the two sides of the market, i.e., the firms are large and matched with many workers. Firm I regards the two types of workers as complements. From employing type 1 workers of mass $x_1 \geq 0$ and type 2 workers of mass $x_2 \geq 0$, firm I derives utility $\min\{x_1, x_2\}$. Firm II wants to hire more workers up to mass of 1 and prefers type 1 workers. Type 1 workers prefer working for firm I, and type 2 workers prefer working for firm II. We can verify that the following three matchings are not in the core just as in Example 2. First, firm I hiring all type 1 and type 2 workers is not in the core since it is blocked by firm II and type 2 workers. Second, firm I hiring no workers and firm II hiring all type 2 workers is not in the core since it is blocked by firm II and type 1 workers. Third, firm I hiring no workers and firm II hiring all type 1 workers is not in the core since it is blocked by the coalition formed by firm I, type 1 workers, and type 2 workers. However, note that the $(1/2, 1/2, 1/2)$ combination of the three matchings above, i.e., the matching in which both firms hire half of the type 1 workers and half of the type 2 workers, is in fact in the core. To verify this, note that although those type 1 workers working for firm II wish to go to firm I, firm I would not benefit from that unless some more type 2 workers also go to firm I. However, type 2 workers have no incentive to do so. On the other hand, although those type 2 workers working for firm I wish to go to firm II, Firm II has no incentive to accept them at the cost of dismissing some type 1 workers.

In this paper, we provide a unified explanation of the three nonempty core examples discussed above: the continuum roommate problem (Example 3), the time-share roommate problem (Example 5), and the large-firm labor market problem (Example 6). In those three examples, we observe a striking similarity: a matching in the core can be obtained by combining a few matchings not necessarily in the core. We introduce a unifying framework, *the convex matching game*, which subsumes a large class of matching problems. We call a matching problem a convex matching game if it satisfies two conditions: (1) The set of all matchings is a convex set; (2) a convex combination of matchings preserves the welfare properties of its components. In all convex matching games, we show that a matching in the core can always be obtained by taking convex combinations; therefore, the core is nonempty. However, note that our concept of convexity is not in its usual sense because the weights a combination assigns to its components do not necessarily sum to 1 as in the three examples. We will see that the weights instead need to satisfy a more subtle constraint.

Scarf's lemma is central to our result, which distinguishes our approach from the fixed-point approach to stable matchings prevalent in the matching theory literature (see Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield and Kominers (2017), Azevedo and Hatfield (2018), and Che, Kim, and Kojima (2019), among others). The lemma first appeared in the seminal paper by Scarf (1967), where it is used to show that a balanced nontransferable utility (NTU) game always has a nonempty core. Later, Aharoni and Fleiner (2003), in their combinatorial mathematics paper, use Scarf's lemma to show the existence of a fractional stable matching in hypergraphic preference systems and highlight the relation of Scarf's lemma to stable matching problems in the sense of Gale and Shapley (1962), although they provide no specific economic interpretation of fractional stable matchings. More recently, Biró, Fleiner, and Irving (2016) apply Scarf's lemma to the hospital / resident matching problem with couples and show the existence of a fractional stable matching, and Nguyen and Vohra (2018) show that a rounding algorithm can be applied to a fractional stable matching to obtain an integral stable matching of a nearby problem with adjusted hospital capacities.

We show that the Scarf's lemma approach can be applied to a large class of matching problems beyond labor market matching with couples, since the concept of convex matching games we introduce allows for general matching networks, multilateral contracts, and complementary or nonconvex preferences. Specifically, we provide three applications:

1. The first application is a full-fledged generalization of the continuum roommate problem (Example 3), which considers a market with a continuum of agents that form social or economic relationships, such as roommateship, marriage, employment, or school enrollment. We follow the tradition of Hatfield and Milgrom (2005) and call these relationships "contracts". The contracts are allowed to be multilateral (involving more than two agents), but each contract is small in the sense that the set of agents involved in each contract is of zero mass. Each agent has arbitrary preferences over sets of contracts that involve her. We show that this problem satisfies the notion of convex matching games and therefore has a nonempty core. This result can be viewed as an alternative approach to the nonempty core result of Kaneko and Wooders (1986) and Azevedo and Hatfield (2018).

2. The second application is a full-fledged generalization of the time-share roommate problem (Example 5), which considers a market with finitely many agents who coordinate through contracts. Each contract is specified by a set of terms (e.g., price, quantity, time, location) that describe what each agent involved in this contract is supposed to do. It is assumed that the contract terms are taken from a convex set and the agents have convex preferences over contract terms. We show that this problem satisfies the notion of convex

matching games and thus has a nonempty core. This result extends the nonempty core result of Hatfield and Kominers (2015) to markets that may not have quasi-linear transfers.

3. The third application is a full-fledged generalization of the large-firm labor market problem (Example 6), which considers a market in which finitely many firms are matched to a continuum of workers of finitely many types and workers may have preferences over their colleagues. We show that this model satisfies the notion of convex matching games and thus has a nonempty core, if all firms and workers have convex preferences and each worker dislikes other workers of the same type possibly due to competition. This setting can be viewed as a generalization of Che, Kim, and Kojima (2019) to allow for peer preferences.

In one-to-one matching problems such as the marriage or roommate problem, it is known that the solution concept of the core is equivalent to stability. In more general matching markets, however, the two solution concepts are not related in a straightforward way. Blair (1988), for example, provides an example of a many-to-many matching problem with a unique stable matching and a different unique matching in the core. As a consequence, the nonempty core results in the first and third applications do not directly imply the existence of stable matchings in Azevedo and Hatfield (2018) and Che, Kim, and Kojima (2019). However, we go one step further to show that in labor markets with a continuum of workers without peer preferences, either in the small-firm setting of Azevedo and Hatfield (2018) or in the large-firm setting of Che, Kim, and Kojima (2019), the notion of convex matching games can be applied to establish the existence of stable matchings despite the difficulty of complementary preferences. In this sense, the framework of convex matching games unifies our understanding of the two similar but distinct results on two-sided large-market matchings with complementarities.

The remainder of this paper is organized as follows. Section 2 discusses the relation of this paper to the literature. Section 3 introduces the framework of convex matching games. Section 4 studies the three applications, and in particular, the third application provides a new nonempty core result for many-to-one matching problems with peer preferences. Section 5 provides a unified understanding of the stability results in Azevedo and Hatfield (2018) and Che, Kim, and Kojima (2019) in the special case of labor markets with a continuum of workers of finitely many types. Section 6 concludes the paper.

2 Literature

As discussed in the Introduction, a lemma of Scarf (1967) plays a central role in our nonempty core result. We show that the Scarf's lemma approach can be applied to a large class of problems beyond matching with couples studied by Biró, Fleiner, and Irving (2016) and Nguyen and Vohra (2018). This section provides a brief literature review for the three applications in this paper.

The first application can be viewed as an alternative approach to the nonempty core result of Azevedo and Hatfield (2018) and Kaneko and Wooders (1986) with some technical extensions. Kaneko and Wooders (1986) show that the core is nonempty in general nontransferable utility (NTU) games with a continuum of agents of finitely many types. A limitation of this result is that they consider a weakened notion of the core that only rules out blocking by coalitions of finitely many agents. By contrast, Azevedo and Hatfield (2018) consider the standard notion of the core while making the assumption that each contract only contains finitely many agents. In this application, we extend the result of Azevedo and Hatfield (2018) to allow for contract terms to vary continuously. Although the nonempty core result does not imply stability in general, in Section 5.1 the Scarf's lemma approach will be applied to obtain the stability result of Azevedo and Hatfield (2018) in the special case many-to-one matching with complementarities.

The second application generalizes the nonempty core result of Hatfield and Kominers (2015) to markets that may not have quasi-linear transfers. Hatfield and Kominer (2015) show that the competitive equilibrium always exists in a multilateral matching setting with convex preferences. Because a competitive equilibrium must be in the core, the nonempty core result is obtained as their corollary. In their model, participants in a venture can jointly determine their level of participation and the monetary transfer to each participant in the venture. We extend this setting to allow participants in a venture to determine all parameters that describe what each of them is supposed to do in the venture, including price, quantity, time, location, etc. In this multilateral matching setting, note that the notions of the core and stability are not related in a straightforward way, and therefore a caveat is that our nonempty core result does not imply the (strong group) stability result of Hatfield and Kominers (2015).

The third application extends the large-firm labor market setting of Che, Kim, and Kojima (2019) to allow for peer preferences. Dutta and Massó (1997) study matching problems with peer preferences and find that the core tends to be empty unless workers have *lexicographic* preferences, taking the firm they work for as their primary consideration and the colleagues they work with as their secondary consideration. This assumption essentially requires that peer preferences are negligible. By contrast, the Scarf's lemma approach allows us to consider peer preferences that are significant because our nonempty core result only relies on convexity of preferences and the assumption that each worker dislikes other workers of the same type. In addition, peer preferences also relate the matching problem to the coalition formation literature (e.g., Banerjee, Konishi, and Sönmez (2001), Cechlárová and Romero-Medina (2001), Bogomolnaia and Jackson (2002), Pápai (2004), and Pycia (2012)), where nonempty core results are typically obtained under relatively restrictive assumptions on agents' preferences or on the contracting network structure.

The framework of convex matching games also unifies our understanding of the stability results of Azevedo and Hatfield (2018) and Che, Kim, and Kojima (2019) in the special case of labor markets with finitely many types of workers. These two papers both study two-sided large matching markets and obtain stability results despite the difficulty of complementary preferences. However, their contributions to the literature are independent since their results are obtained in different model settings. Azevedo and Hatfield (2018) consider a model with a continuum of agents on both sides of the market and where each agent is matched to finitely many agents on the other side.[3] By contrast, Che, Kim, and Kojima (2019) consider a model with finitely many firms, each of which is matched to a continuum of workers.[4] As a consequence, the stability results in the two papers correspond to different asymptotic stability properties of large finite markets: The model of Azevedo and Hatfield (2018) can be interpreted as the limit of a sequence of markets in which number of firms and the number of workers go to infinity at the same speed while the size of each firm remains unchanged; the model of Che, Kim, and Kojima (2019) can be interpreted as the limit of a sequence of markets in which the number of firms remains unchanged while the number of workers and the size of each firm go to infinity at the same speed.

3 Model

This section introduces the concept of convex matching games and shows that a convex matching game always has a nonempty core.

[3]Studies that consider the "small-firm" setting as in Azevedo and Hatfield (2018) include Echenique, Lee, Shum, and Yenmez (2013) and Menzel (2015), among others.

[4]Studies that consider the "large-firm" setting as in Che, Kim, and Kojima (2019) include Abdulkadiroğlu, Che, and Yasuda (2015) and Azevedo and Leshno (2016), among others.

A *matching game* consists of $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$. The finite set I is the set of *players* and $\mathcal{M}$ is the set of *matching outcomes*, or *matchings* for short. In applications, a player may represent either a single agent or a continuum of identical agents. The function $\phi : \mathcal{M} \to [0,1]^I$ is called the *participation function*. For each matching μ, the value $\phi_i(\mu) \in [0,1]$ measures the participation of player i in matching μ. When player i represents a single agent, $\phi_i(\mu)$ is either 0 or 1, indicating whether agent i participates in matching μ. When player i represents a continuum of identical agents, $\phi_i(\mu)$ is the fraction of type i agents participating in matching μ. For each player $i \in I$, let $\mathcal{M}_i := \{\mu \in \mathcal{M} : \phi_i(\mu) > 0\}$ be the set of matchings in which player i has a positive participation level. The binary relation $\trianglerighteq_i$ on $\mathcal{M}_i$ is the *preference relation* of player i, which is assumed to be complete and transitive. When player i represents a continuum of identical agents, the preference relation $\trianglerighteq_i$ is typically interpreted as aggregate preferences, determined by the welfare of the worst-off type i agents. In other words, $\mu' \trianglerighteq_i \mu$ means that the worst-off type i agents under matching μ' are better off than the worst-off type i agents under matching μ.[5] Let $\triangleright_i$ be the strict version of $\trianglerighteq_i$ and $\equiv_i$ be indifference. The binary relation $\sqsupset$ on $\mathcal{M}$ is interpreted as the *blocking relation*. When $\mu' \sqsupset \mu$, we say that matching μ' blocks matching μ. A matching μ is *nontrivial* if $\phi(\mu^0) \neq \mathbf{0}$, which means that μ is not the matching under which all agents are unmatched.

The *core* of the matching game is the set of matchings that are not blocked by any nontrivial matching. The formal definition is below.

Definition 1. The *core* of a matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$ is the set

$$C := \bigcap_{\mu \in \mathcal{M}_+} NB(\mu)$$

where $\mathcal{M}_+$ is the set of all nontrivial matchings and $NB(\mu) := \{\mu' \in \mathcal{M} : \mu \not\sqsupset \mu'\}$ is the set of matchings not blocked by μ.

To better understand the concept of matching games introduced above, let us consider how it relates to the three examples in the introduction. First, in the continuum roommate problem (Example 3), there are three players, each representing a continuum of individuals of the same type, i.e., we have $I = \{A, B, C\}$. Let μ^1 be the matching in which all type A individuals are matched with type B individuals and all type C individuals are unmatched,

[5]When dealing with stability instead of the core, the interpretation of the relation $\trianglerighteq_i$ will be modified accordingly. See Section 5for details.

μ^2 be the matching in which all type B individuals are matched with type C individuals and all type A individuals are unmatched, and μ^3 be the matching in which all type C individuals are matched with type A individuals and all type B individuals are unmatched. Furthermore, let μ^* be the matching in which half of the type A individuals are matched with half of the type B individuals, half of the type B individuals are matched with half of the type C individuals, and half of the type C individuals are matched with half of the type A individuals. The set $\mathcal{M}$ contains the set of all matchings, including but not restricted to μ^1, μ^2, μ^3, and μ^*. The participation vectors of these particular matchings are $\phi(\mu^1) = (1, 1, 0)$, $\phi(\mu^2) = (0, 1, 1)$, $\phi(\mu^3) = (1, 0, 1)$, and $\phi(\mu^*) = (1, 1, 1)$. Note that the worst-off type A individuals in μ^1 are matched to a type B individual, while the worst-off type A individuals in μ^* are matched to a type C individual. Therefore, we have $\mu^1 \rhd_A \mu^*$. Moreover, we have $\mu^3 \equiv_A \mu^*$ since the worst-off type A individuals are matched to a type B individual in both matchings. Note that $\mu^2 \notin \mathcal{M}_A$ and so it is not comparable under $\trianglerighteq_A$. The relations $\trianglerighteq_B$ and $\trianglerighteq_C$ are obtained in a similar way. As in the Introduction, we have $\mu^2 \sqsupset \mu^1$, $\mu^3 \sqsupset \mu^2$, and $\mu^1 \sqsupset \mu^3$, and the matching μ^* is not blocked by any nontrivial matching.

Second, in the time-share roommate problem (Example 5), there are three players, individuals A, B, and C, i.e., we have $I = \{A, B, C\}$. Let μ^1 be the matching in which individuals A and B are always matched while individual C is unmatched. Let μ^2 be the matching in which individuals B and C are always matched while individual A is unmatched. Let μ^3 be the matching in which individuals C and A are always matched while individual B is unmatched. Let μ^* be the matching in which each pair of individuals are matched together half of the time. The participation vectors of these particular matchings are $\phi(\mu^1) = (1, 1, 0)$, $\phi(\mu^2) = (0, 1, 1)$, $\phi(\mu^3) = (1, 0, 1)$, and $\phi(\mu^*) = (1, 1, 1)$. By the preferences of individual A, we have $\mu^1 \rhd_A \mu^* \rhd_A \mu^3$. Note that $\mu^2 \notin \mathcal{M}_A$ and so it is not comparable under $\trianglerighteq_A$. The relations $\trianglerighteq_B$ and $\trianglerighteq_C$ are obtained in a similar way. As in the Introduction, we have $\mu^2 \sqsupset \mu^1$, $\mu^3 \sqsupset \mu^2$, and $\mu^1 \sqsupset \mu^3$, and the matching μ^* is not blocked by any nontrivial matching.

Finally, in the large-firm labor market problem (Example 6), there are four players: firm I, firm II, type 1 workers, and type 2 workers, i.e., $I = \{I, II, 1, 2\}$. Note that I and II each represent a single firm, while 1 and 2 each represent a continuum of identical workers. Let μ^1 be the matching in which firm I hires all workers. Let μ^2 be the matching in which firm I hires no workers and firm II hires all type 2 workers. Let μ^3 be the matching in which firm I hires no workers and firm II hires all type 1 workers. Let μ^* be the matching in which both firms hire half of the workers of each type. The participation vectors of these particular matchings are $\phi(\mu^1) = (1, 0, 1, 1)$, $\phi(\mu^2) = (0, 1, 0, 1)$, $\phi(\mu^3) = (0, 1, 1, 0)$, and $\phi(\mu^*) = (1, 1, 1, 1)$. By the preferences of the firms, we have $\mu^1 \rhd_I \mu^*$ and $\mu^3 \rhd_{II} \mu^* \rhd_{II} \mu^2$. By considering the worst-off workers within a type, we have $\mu^1 \rhd_1 \mu^3 \equiv_1 \mu^*$ and $\mu^2 \rhd_2 \mu^1 \equiv_1 \mu^*$. As in the Introduction, we have $\mu^2 \sqsupset \mu^1$, $\mu^3 \sqsupset \mu^2$, and $\mu^1 \sqsupset \mu^3$, and the matching μ^* is not blocked by any nontrivial matching.

In all three examples above, the matching μ^* in the core can be considered a $(1/2, 1/2, 1/2)$ combination of μ^1, μ^2, and μ^3 as we have saw the Introduction. Now let us formally introduce the notion of combinations of matchings. Let μ^1, μ^2, ..., and μ^n be a set of matchings in $\mathcal{M}$ and $w = (w^1, w^2, \ldots, w^n)$ be a vector of nonnegative real numbers. We assume that the set $\mathcal{M}$ of all matchings is endowed with some algebraic structure s.t. the linear combination $\sum_{j=1}^n w^j \mu^j$ also represents a matching in $\mathcal{M}$ as long as $\sum_{j=1}^n w^j \phi(\mu^j) \leq \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^I$ is the vector of 1's. This structure is natural in the three examples. In the continuum roommate problem (Example 3), a linear combination of matchings scales the mass of individuals matched under μ^j by w^j and then pools all the scaled matchings together. In the time-share roommate problem (Example 5), a linear combination of matchings specifies the time each pair of individuals spend together as the corresponding linear combination of the time they spent together in each μ^j. In the large-firm labor market problem (Example 6), a linear combination of matchings specifies the type distribution of each firm's employees as the corresponding linear combination of type distributions in each μ^j. The restriction $\sum_{j=1}^n w^j \phi(\mu^j) \leq \mathbf{1}$ is required because in the first and third examples, the fraction of matched individuals or hired workers of each type cannot be greater than 1, and in the second example, each individual has 1 unit of available time. Let us call the linear combination $\sum_{j=1}^n w^j \mu^j$ a *ϕ-convex combination* if the weights satisfy the restriction $\sum_{j=1}^n w^j \phi(\mu^j) \leq \mathbf{1}$. Note that this is not the standard notion of convex combinations since the weights (w^j) do not necessarily sum to 1. Instead, the restriction is determined by the participation function ϕ.

The ϕ-convex structure of the set of matchings by itself is not sufficient for a nonempty core. It is also important to assume that a ϕ-convex combination of matchings preserves the welfare properties of its components in some way. Specifically, we assume that a ϕ-convex combination is not blocked by a matching μ if there exists a player i whose participation is necessary to form block μ but has no incentive to so because player i has full participation in more preferred matchings under the ϕ-convex combination. More formally, we impose the following restriction on the blocking relation $\sqsupset$: If μ is a matching, $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-

convex combination and there exists a player i with $\phi_i(\mu) > 0$ s.t. $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, then we must have $\mu \not\sqsupset \mu^*$. This restriction is natural in the three examples we have discussed. First, in the continuum roommate problem, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$ implies that all type i individuals are matched under μ^*. Furthermore, recall that the aggregate preference relation $\trianglerighteq_i$ is determined by the worst-off type i individuals under two matchings, the condition $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$ implies that the worst-off type i individual under μ^* is weakly better-off than the worst-off type i individual under μ. Therefore, there is no type i individual willing to accept the least preferred roommate under the block μ, and so μ does not block μ^*. Second, in the time-share roommate problem, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$ implies that $\sum_{j:\phi_i(\mu^j)>0} w^j = 1$. Therefore, the time individual i spends with another individual under μ^* is a convex combination (in its standard meaning) of that under each μ^j. Moreover, the condition $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$ implies that individual i weakly prefers each relevant component of μ^* to μ. As long as individual i has convex preferences over time spent with others, she will weakly prefer μ^* to μ, and so μ does not block μ^*. Third, in the large-firm labor market problem, when player i represents a type of workers, the same argument carries over as in the first example, and when player i represents a firm, the same argument carries over as in the second example, with the time spent with others being replaced by the type distribution of employees.

Let us summarize the restrictions we impose on a matching game discussed above by defining the notion of *convex matching games*.

Definition 2. A matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$ is *convex* if (i) the set $\mathcal{M}$ of matchings is closed under ϕ-convex combinations and (ii) we have $\mu \not\sqsupset \mu^*$ if $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ with $\phi_i(\mu) > 0$ s.t. $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$ and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$.[6]

In addition to the convexity restriction, we also need the following technical restriction.

Definition 3. A matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$ is *regular* if (i) the set $\mathcal{M}$ of matchings is compact and (ii) for each nontrivial matching μ, the set $NB(\mu)$ of matchings that are not blocked by μ is closed.[7]

[6]Convexity of matching games implicitly requires that $\mathcal{M}$ lies in a vector space so that linear combinations are meaningful.

[7]Regularity of matching games implicitly requires that $\mathcal{M}$ lies in a topological space so that compactness and closedness are meaningful

Here is the main result of this paper.

Theorem 1. *The core is nonempty in a regular and convex matching game.*

Central to this result is a lemma by Scarf (1967). The following version of the lemma can be found in, for example, Király and Pap (2009). Consider a matrix $A = (a_{ij})$ of nonnegative real numbers with rows indexed by i in a finite set I and columns indexed by $j \in \{1, 2, \ldots, n\}$. Assume that in each column j, at least one entry is positive. Each row i is associated with a complete and transitive binary relation $\trianglerighteq_i$ on the set $\{j : a_{ij} > 0\}$ of columns.[8] We say that a vector w in the polyhedron $\{w \in \mathbb{R}^n_+ : Aw \leq \mathbf{1}\}$ *dominates* column $k \in \{1, 2, \ldots, n\}$, if there exists a row i s.t. $a_{ij} > 0$, $\sum_{j=1}^n w^j a_{ij} = 1$, and $j \trianglerighteq_i k$ for all j with $w^j > 0$ and $a_{ij} > 0$.

Lemma 1 (Scarf, 1967)**.** *There exists a vector w^* in the polyhedron $\{w \in \mathbb{R}^n_+ : Aw \leq \mathbf{1}\}$ that dominates all columns of the matrix A.*

Proof of Theorem 1. Consider a regular and convex matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$. To show that the core

$$C = \bigcap_{\mu \in \mathcal{M}_+} NB(\mu)$$

is nonempty, it is sufficient to show that every finite collection of these $NB(\mu)$'s has a nonempty intersection. To see the sufficiency here, suppose that $C = \bigcap_{\mu\in\mathcal{M}_+} NB(\mu)$ is empty. Then, the complements of the $NB(\mu)$s consist of an open cover of $\mathcal{M}$ because each $NB(\mu)$ is closed. By the compactness of $\mathcal{M}$, there exists a finite subcover, which corresponds to finitely many $NB(\mu)$s with empty intersection.

Take any finite collection of nontrivial matchings: $\mu^1, \mu^2, \ldots, \mu^n \in \mathcal{M}_+$. We want to show that the intersection $\bigcap_{k=1}^n NB(\mu^k)$ is nonempty. Let us construct the $|I| \times n$ matrix A with its j-th column being $\phi(\mu^j)$, i.e., we let $a_{ij} = \phi_i(\mu^j)$. Note that each column j of A contains at least one positive entry because μ^j is nontrivial. Moreover, because the aggregate preference relation $\trianglerighteq_i$ of player i is defined on all matching μ's with $\phi_i(\mu) > 0$, it is well defined on all column js with $a_{ij} > 0$. Therefore we can invoke Scarf's lemma to obtain a weight vector w^* that dominates all columns. By part (i) of Definition 2, the

[8] Király and Pap (2009) and Nguyen and Vohra (2018) require the relation $\trianglerighteq_i$ to be a linear order, which rules out indifference. However, it is not difficult to accommodate indifference. If $\trianglerighteq_i$ does not satisfy antisymmetry, we may break ties arbitrarily to obtain a linear order $\trianglerighteq_i'$. By applying Scarf's lemma to $\trianglerighteq_i'$, we obtain a dominating vector with respect to $\trianglerighteq_i'$ and note that a dominating vector with respect to $\trianglerighteq_i'$ is also a dominating vector with respect to the original relation $\trianglerighteq_i$.

ϕ-convex combination $\mu^* = \sum_{j=1}^{n} w^{*j}\mu^j$ is a matching in $\mathcal{M}$. Moreover, because w^* dominates each column k, by part (ii) of Definition 2, the matching μ^* is not blocked by μ^k. This shows that $\mu^* \in \bigcap_{k=1}^{n} NB(\mu^k)$, and therefore the proof is completed. $\square$

4 Applications

This section discusses three applications of the concept of convex matching games, generalizing each of the three nonempty core examples in the Introduction.

4.1 Continuum Economy with Small Contracts

In this subsection, we consider a continuum economy model in which each contract only involves a set of agents of zero mass and show that it satisfies our notion of convex matching games and therefore has a nonempty core. This model subsumes the continuum roommate problem (Example 3) as a special case and can be viewed as a technical extension of the nonempty core result of Azevedo and Hatfield (2018) to a setting where a contract may involve a continuum of agents and a continuum of contract terms. An important feature of this model is that it allows for general matching networks, multilateral contracts, and complementary or nonconvex preferences, which we know usually lead to the empty core issue in finite markets.

Consider finitely many types of agents, denoted by $i \in I$. The mass of type i agents is $m_i > 0$. Agents play *roles* in social or economic relationships with other agents. We let R_i be the set of roles for type i agents and assume that each R_i is a compact metric space. Each social or economic relationship is a combination of roles. We follow the tradition of Hatfield and Milgrom (2005) and call social or economic relationships *contracts*. We represent a *type of contracts* as $x = \{x_i\}_{i\in I}$, where x_i is a Borel measure on R_i that specifies the quantity of each role for type i agents involved in the contract. Let X be the set of all types of contracts, and we assume that X is compact.[9] In addition, we exclude from X the "empty" contract, i.e., the contract x with each x_i being the zero measure on R_i.

Each agent may participate in multiple contracts at the same time, playing one role in each contract. For a type i agent, a *bundle of roles* is a Borel measure β_i on R_i that only

[9]In this paper, the topology we adopt is the weak-* topology whenever we consider a set of measures. In the context of the space X, the weak-* topology is the weakest topology that makes $\int_{R^i} f dx_i$ a continuous function in x for each i and continuous f on R^i.

takes integer values, and it specifies the quantity of each role contained in the bundle. We assume that for some finite integer N, each agent cannot take more than N roles at a time, i.e., the bundle β_i is bounded by N. Let $\bar{\mathfrak{B}}_i$ be the set of all such bundles. For each agent type i, agents of type i share the same preference relation $\succsim_i$ on $\bar{\mathfrak{B}}_i$, which is assumed to be complete and transitive. Moreover, we assume that the preference relation $\succsim_i$ is continuous in the sense that all upper or lower contour sets are closed in $\bar{\mathfrak{B}}_i$. Let $\mathbf{0}_i \in \bar{\mathfrak{B}}_i$ be the zero measure on R_i, which represents the empty bundle. When a type i agent holds the empty bundle, we also say that the agent is unmatched; otherwise, we say that the agent is matched. Let $\mathfrak{B}_i := \{\beta_i \in \bar{\mathfrak{B}}_i \backslash \{\mathbf{0}_i\} : \beta_i \succsim_i \mathbf{0}_i\}$ be the set of nonempty and individually rational bundles for type i agents.

A matching outcome, or simply a *matching*, is described by a Borel measure μ_x on X that specifies the quantity of contracts of each type x and a Borel measure μ_i on $\mathfrak{B}_i$ that specifies the quantity of type i agents holding each nonempty and individually rational bundle β_i. Clearly, we have $\mu_i(\mathfrak{B}_i) \leq m_i$ since the mass of type i agents is m_i, and the difference $m_i - \mu_i(\mathfrak{B}_i)$ is the mass of unmatched type i agents. In a matching, we require that the following accounting identity must hold for each agent type i:

$$\int_{\mathfrak{B}_i} \beta_i d\mu_i = \int_X x_i d\mu_x. \tag{1}$$

Note that both sides of the identity measure the quantity of each role for type i agents present in the matching μ. The left-hand side calculates the quantity from the perspective of bundles, and the right-hand side calculates the quantity from the perspective of contracts.[10] Now, we formally define a matching as $\mu = (\mu_x, (\mu_i)_{i \in I})$ subject to the accounting identity (1).

Note that we do not assume that each contract only contains finitely many roles. In general, a contract may contain a continuum of roles played by a continuum of agents, as long as the continuum of agents is of zero mass. In that case, our model can be viewed as the limit of a large economy model in which the number of agents involved in each contract grows sublinearly with respect to the number of agents in the whole economy.

[10]Intuitively, both sides of the identity are aggregations of a continuum of Borel measures on R^i, which results in a Borel measure on R^i. Formally, it is easier to define the Borel measures on both sides of the identity indirectly through the linear functional to which they correspond. This can be done since each linear functional L on the set of continuous functions on R^i s.t. $L(f) \geq 0$ if $f \geq 0$ corresponds to a unique Borel measure on R^i. We can define the left-hand side of the identity as the Borel measure that corresponds to the linear functional $L(f) := \int_{\mathfrak{B}_i} \int_{R^i} f d\beta_i d\mu_i$ and the right-hand side as the Borel measure that corresponds to the linear functional $L'(f) := \int_X \int_{R^i} f dx_i d\mu_x$.

The *trivial matching* is the one with μ_x being the zero measure on X and μ_i being the zero measure on $\mathfrak{B}_i$, i.e., the matching in which all agents are unmatched. For a matching μ with $\mu_i(\mathfrak{B}_i) > 0$, we let $\underline{\beta}_i(\mu)$ be the least preferred nonempty bundle held by type i agents under matching μ.[11] We say that a matching μ is blocked by a nontrivial matching $\hat{\mu}$ if for any agent type i involved in block $\hat{\mu}$, we are able to find some type i agents willing to participate in the block. The formal definition is as below.

Definition 4. A matching μ is *blocked* by a nontrivial matching $\hat{\mu}$ if for each $i \in I$ with $\hat{\mu}_i(\mathfrak{B}_i) > 0$, we have

$$m_i - \mu_i(\mathfrak{B}_i) + \mu_i(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\hat{\mu})\}) > 0.$$

The *core* of this continuum economy is the set of matchings that are not blocked by any nontrivial matching.

In the inequality in definition, $m_i - \mu_i(\mathfrak{B}_i)$ is the mass of type i agents who are unmatched and $\mu_i(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\hat{\mu})\})$ is the mass of type i agents who are matched, but the bundle they hold is less preferred than the least preferred bundle $\underline{\beta}_i(\hat{\mu})$ under $\hat{\mu}$. Therefore the left-hand side of the inequality measures the total mass of type i agents willing to accept $\underline{\beta}_i(\hat{\mu})$. The inequality requires this mass to be positive since we need to find some type i agents willing to accept $\underline{\beta}_i(\hat{\mu})$ to form such a block $\hat{\mu}$. Additionally, we may also require that for each bundle β_i in the support of $\hat{\mu}_i$, the mass of type i agents willing to accept β_i is no less than the mass of type i agents holding some bundle weakly worse than β_i under $\hat{\mu}$. However, this additional requirement will not make a difference to the notion of the core because as long as the mass of type i agents willing to accept $\underline{\beta}_i(\hat{\mu})$ is positive, we can find $\alpha > 0$ small enough s.t. the mass of type i agents willing to accept $\underline{\beta}_i(\hat{\mu})$ is greater than $\alpha\hat{\mu}_i(\mathfrak{B}_i)$, which implies that there are sufficiently many type i agents willing to accept any bundle under the block $\alpha\hat{\mu}$.

Note that the model implicitly assumes "small contracts" in the sense that agents are coordinated through a continuum of contracts, each of which only involves a set of agents of zero mass. This rules out the large coalition formation problem (Example 4). Under this assumption, each matching μ is perfectly divisible, and more importantly, the support

[11] Formally, we define $\underline{\beta}_i(\mu)$ as the least preferred bundle by type i agents in the support of μ_i. Note that the least preferred bundle exists because the support of μ_i is by definition closed, $\mathfrak{B}_i$ is compact (see Proof of Proposition 1 in Appendix A), and the preference relation $\succsim_i$ is continuous. If there is more than one least preferred bundle due to indifference, we arbitrarily let one of them be $\underline{\beta}_i(\mu)$.

of $\alpha\mu_i$ is the same as that of μ_i, so they have the same welfare implications for matched agents. This property is crucial to the convexity structure of the matching game induced by the continuum economy model.

To apply our main theorem (Theorem 1), let us consider the following matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$ induced by the continuum economy. Let the set I of players be the set of agent types and the set $\mathcal{M}$ be the set of matchings. For each matching $\mu = \{\mu_x, \{\mu_i\}_{i\in I}\}$, let the participation value $\phi_i(\mu)$ of type i agents be the fraction of them being matched under μ, i.e., $\phi_i(\mu) = \mu_i(\mathfrak{B}_i)/m_i$. For each agent type i, define the aggregate preference $\trianglerighteq_i$ on $\mathcal{M}_i$, the set of matchings with $\mu_i(\mathfrak{B}_i) > 0$, as $\mu' \trianglerighteq_i \mu$ if $\underline{\beta}_i(\mu') \succsim_i \underline{\beta}_i(\mu)$. The relation $\sqsupset$ follows the blocking relation in Definition 4, i.e., we let $\hat{\mu} \sqsupset \mu$ if $\hat{\mu}$ is nontrivial and $\hat{\mu}$ blocks μ.

Proposition 1 (Kaneko and Wooders (1986), Azevedo and Hatfield (2018))**.** *With continuous preferences and small contracts, the matching game induced by the continuum economy model is regular and convex. Therefore, the core of the continuum economy is nonempty by Theorem 1.*

Proof. See Appendix A. □

As we have mentioned, this nonempty core result is obtained in a general setting that allows for general matching networks, multilateral contracts, and complementary or non-convex preferences. To better understand the generality of the setting, let us for example consider a labor market matching problem with couples and peer preferences. There is a continuum of firms categorized into finitely many types, which is defined by the firms' preferences, size, industry, market share, financial structure, etc. There is also a continuum of families, each of which has a male worker and a female worker. Families are also categorized into finitely many types defined by the two workers' preferences, skill type, ability, experience, etc. Each firm takes into account the characteristics of all its employees, and its preferences may exhibit complementarities over different skill types. The two workers in each family make their joint decision, taking into account not only the characteristics of the firms they work for but also the characteristics of their colleagues and the distance between the firms they work for. Of course, both firms and workers also care about other characteristics of the employment relationship, including wage, working conditions, etc. In our model, let us regard the employment relationship between a firm and all its employees' families as one contract. Each contract type x specifies the type of firm and the number of each type of employees' families that supply to the contract a male worker, a female worker, or both under each wage level and each set of working conditions. The role the firm plays in a type x contract is simply "the employer of a type x contract". The role played by a family in a type x contract is specified by the type of the family, whether the family is a supplier of a male worker, a female worker, or both, and the wage and working conditions provided to the family.[12] Each firm may play at most one role at a time, and a family may play at most two roles, a male worker supplier in one contract and a female worker supplier in another, in which case the two workers in the family work for different firms.

4.2 Finite Economy with Convex Set of Contract Terms

In this subsection, we consider a finite economy model in which each contract is specified by a term chosen from a convex set. This model subsumes the time-share roommate problem as a special case and is a generalization of a nonempty core result of Hatfield and Kominers (2015) to markets that may not have quasi-linear transfers. We show that if all agents have convex preferences over contract terms, the model satisfies our notion of convex matching games and therefore has a nonempty core.

There is a finite set I of agents and a finite set Ω of *ventures*. For each venture $\omega \in \Omega$, let $a(\omega) \subset I$ be the nonempty set of participants in the venture, who jointly determine the contract terms μ_ω that specify how the participants interact with one another in the venture. Let us assume that the vector μ_ω of contract terms is chosen from some convex and compact set $\mathcal{M}_\omega \subset \mathbb{R}^{N_\omega}$.

Consider a bilateral trading network for example. Each venture ω is a trading relationship between two agents, i.e., $a(\omega) = \{i, i'\}$. A typical vector μ_ω of contract terms may be $(3, 5, -7)$, which represents "i gives 3 apples and 5 bananas to i', and i' gives 7 dollars to i in return". Each agent may be involved in multiple trading relationships with other agents, since the model allows the sets of participants of two ventures to overlap or even coincide. More generally, a vector of contract terms may include price, quantity, time, location, etc. We assume that the zero vector $\mathbf{0}_\omega$ of $\mathbb{R}^{N_\omega}$ is in $\mathcal{M}_\omega$, which represents the state of having no interaction among participants in the venture ω. We say that the venture ω is active if

[12]This example explains why the "role" r is not a redundant concept in the model given that we already have the notion "contract type" x. In this example, it is clear that a type x contract may treat the families involved in the contract differently even if the families are of the same type. In the same contract, one family might be a supplier of the female worker under a high wage while another family might be a supplier of both workers under a low wage, which makes a significant difference to the families' welfare. It is therefore important to use the concept of "roles" to keep track of the well-being of each family.

$\mu_\omega \neq \mathbf{0}_\omega$.

A matching is described by the contract terms of each venture. Formally, a matching is $\mu = \{\mu_\omega\}_{\omega\in\Omega}$. Let $\bar{\mathcal{M}} := \prod_{\omega\in\Omega} \mathcal{M}_\omega$ be the set of all matchings. For each agent i, let $\Omega_i := \{\omega \in \Omega : i \in a(\omega)\}$ be the set of all ventures that include i as a participant. Agent i has a complete and transitive preference relation $\succsim_i$ on $\prod_{\omega\in\Omega_i} \mathcal{M}_\omega$, the contract terms in all ventures ω with $i \in a(\omega)$. We assume that $\succsim_i$ is convex in the sense that the upper contour set $\{\{\mu_\omega\}_{\omega\in\Omega_i} \in \prod_{\omega\in\Omega_i} \mathcal{M}_\omega : \{\mu_\omega\}_{\omega\in\Omega_i} \succsim_i \{\mu'_\omega\}_{\omega\in\Omega_i}\}$ is convex for all $\{\mu'_\omega\}_{\omega\in\Omega_i} \in \prod_{\omega\in\Omega_i} \mathcal{M}_\omega$. In addition, we also assume that $\succsim_i$ is upper semi-continuous in the sense that each upper contour set is closed. We can naturally extend the domain of the preference relation $\succsim_i$ to $\bar{\mathcal{M}}$ s.t. $\mu \succsim_i \mu'$ if $\{\mu_\omega\}_{\omega\in\Omega_i} \succsim_i \{\mu'_\omega\}_{\omega\in\Omega_i}$. Let us denote by $\mathbf{0}$ the trivial matching in $\bar{\mathcal{M}}$, i.e., the matching with $\mu_\omega = \mathbf{0}_\omega$ for all ventures $\omega \in \Omega$. Let $\mathcal{M}$ be the set of all individually rational matchings in $\bar{\mathcal{M}}$, i.e., $\mathcal{M} := \{\mu \in \bar{\mathcal{M}} : \mu \succsim_i \mathbf{0} \text{ for all } i \in I\}$. Because only individually rational matchings are relevant to our analysis of the core, in the following discussion, a matching always refers to an individually rational matching.[13]

We say that agent i participates in a matching if there exists at least one active venture that includes i as a participant. Let $\mathcal{M}_i$ be the set of matchings in $\mathcal{M}$ that agent i participates in, i.e., we define $\mathcal{M}_i := \{\mu \in \mathcal{M} : \exists\, \omega \in \Omega_i \text{ s.t. } \mu_\omega \neq \mathbf{0}_\omega\}$. Let $a(\mu) := \{i \in I : \mu \in \mathcal{M}_i\}$ be the set of participants of the matching μ. A matching μ is blocked by a nontrivial matching $\hat{\mu}$ if all participants of block $\hat{\mu}$ are willing to participate in the block. The formal definition is provided below.

Definition 5. A matching μ is *blocked* by a nontrivial matching $\hat{\mu}$ if $\hat{\mu} \succ_i \mu$ for all $i \in a(\mu)$. The *core* of this economy is the set of matchings that are not blocked by any nontrivial matching.

To apply our main theorem (Theorem 1), let us consider the following matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i\in I}, \sqsupset\}$ induced by this finite economy. Let the set I of players be the set of agents and the set $\mathcal{M}$ be the set of individually rational matchings as defined above. For each matching $\mu \in \mathcal{M}$, let the participation value $\phi_i(\mu)$ of agent i be 1 if $i \in a(\mu)$ and be 0 otherwise. For each agent i, let the relation $\trianglerighteq_i$ be the preference relation $\succsim_i$ restricted to $\mathcal{M}_i$. The relation $\sqsupset$ follows the blocking relation in Definition 5, i.e., we let $\hat{\mu} \sqsupset \mu$ if $\hat{\mu}$ is nontrivial and $\hat{\mu}$ blocks μ.

[13] We have implicitly addressed the feasibility of a matching together with its individual rationality. When an agent finds the combination of contract terms under a matching technologically infeasible, we may alternatively consider the matching to be strictly less preferred by the agent than the trivial matching $\mathbf{0}$. In this case, individual rationality also captures technological feasibility, and the convexity of preferences also captures the convexity of technology set.

Proposition 2. *With convex and upper semi-continuous preferences, the matching game induced by the finite economy model is regular and convex. Therefore, the core of the finite economy is nonempty by Theorem 1.*

Convex preferences are crucial to the nonempty core result for the finite economy. This reminds us of the classical result on the existence of Walrasian equilibrium with convex preferences and production sets (Arrow and Debreu (1954), McKenzie (1954)). Our model subsumes general equilibrium models as a special case by restricting the contract terms of a venture to only specify the transfers of goods and money among its participants. Therefore, Proposition 2 implies a nonempty core in general equilibrium models with convex preferences and technology sets.

4.3 Large-Firm Labor Market with Peer Preferences

This subsection generalizes the model of Che, Kim, and Kojima (2019) to allow for workers' peer preferences. Consider a labor market model with finitely many firms and a continuum of workers of finitely many types. Each firm is large in the sense that it may hire a continuum of workers of positive mass. Workers have preferences not only over firms but also over their colleagues. Note that this model is not a special case of our first application discussed in Section 4.1 because it does not satisfy the small contract assumption. In this model, each contract involves a firm and all its employees, which constitute a significant fraction of the labor market.

We show that this model satisfies our notion of convex matching games and therefore has a nonempty core if (1) each firm has convex preferences over type distributions of its workers, (2) each worker has convex preferences over type distributions of her colleagues, and (3) each worker dislikes colleagues of the same type as hers. The third assumption is more relevant in applications where worker types are identified by skills and so more colleagues with the same skill type implies lower pay and a higher chance of job loss due to competition. For this reason, we call the third assumption *within-type competition*.

Let F be the finite set of firms and Θ be the finite set of worker types. For each worker type $\theta \in \Theta$, let $m(\theta) > 0$ be the mass of type θ workers. The vector $m \in \mathbb{R}^{\Theta}_{++}$ is therefore the type distribution of all workers in the market. A set of workers can be described by its type distribution $x \in \mathbb{R}^{\Theta}_{+}$, where $x(\theta) \in [0, m(\theta)]$ is the mass of type θ workers in the set. Let $X := \prod_{\theta \in \Theta} [0, m(\theta)]$ be the set of all type distributions. A matching μ specifies the set of employees of each firm f. Formally, a matching is defined as $\mu = \{\mu_f\}_{f \in F}$, where $\mu_f \in X$ represents the type distributions of the employees of firm f. Feasibility requires $\sum_{f \in F} \mu_f \leq m$. The difference $m - \sum_{f \in F} \mu_f$ is therefore the type distribution of unemployed workers.

Each firm $f \in F$ has preferences over the type distributions of its workers. Formally, each firm f has a complete and transitive preference relation $\succsim_f$ on X. We assumed that $\succsim_f$ is convex in the sense that the upper contour set $\{x \in X : x \succsim_f x'\}$ is convex for each $x' \in X$. We further assume that $\succsim_f$ is upper semi-continuous in the sense that each upper contour set is closed.

Workers of type θ share the same preferences over firms and type distributions of their colleagues. Formally, type θ workers have a complete and transitive preference relation $\succsim_\theta$ on $(F \times X) \cup \{\emptyset\}$, where a typical alternative $(f, x) \in F \times X$ represents the state of being employed by firm f with a set of colleagues of type distribution x and the special alternative $\emptyset$ represents the state of being unemployed. This setup subsumes no peer preferences as a special case by letting $(f, x) \sim_\theta (f', x')$ whenever $f = f'$. We assume that $\succsim_\theta$ is convex in the sense that for each alternative $a \in (F \times X) \cup \{\emptyset\}$ and each firm $f \in F$, the upper contour set $\{x \in X : (f, x) \succsim_\theta a\}$ is convex. Moreover, we also assume that $\succsim_\theta$ is upper semi-continuous in the sense that each upper contour set is closed.

Additionally, we assume that for each worker type θ and firm f, we have $(f, x) \succsim_\theta (f, x')$ for all $x, x' \in X$ with $x(\theta) = 0$ and $x'(\theta) > 0$, and we call this assumption *within-type competition*. This assumption is relevant when there is strong competition between colleagues of the same type to the extent that a worker's primary consideration regarding her colleagues is whether there are colleagues of the same type as hers, while all other aspects of the type distribution of her colleagues are secondary considerations. Combined with the convexity of $\succsim_\theta$, within-type competition implies that the upper contour set $\{x \in X : (f, x) \succsim_\theta (f, x')\}$ of a type distribution x' with $x'(\theta) > 0$ contains the pyramid whose apex is x' and whose base is the rectangle $\{x \in X : x(\theta) = 0\}$. In particular, the within-type competition assumption implies that if two type distributions x and x' agree on the mass of all worker types other than θ, we have $(f, x') \succsim_\theta (f, x)$ iff $x'(\theta) \leq x(\theta)$.

We say that a matching $\mu = \{\mu_f\}_{f \in F}$ is individually rational if $\mu_f \succsim_f \mathbf{0}$ for each firm f and $(f, \mu_f) \succsim_\theta \emptyset$ for each worker type θ with $\mu_f(\theta) > 0$. Let $\mathcal{M}$ be the set of all individually rational matchings. Because only individually rational matchings are relevant to our analysis of the core, in the following discussion, a matching always refers to an individually rational matching.

A *simple* matching (f, x) refers to a matching μ with $\mu_f = x \neq \mathbf{0}$ and $\mu_{f'} = \mathbf{0}$ for all

$f' \neq f$. A matching is blocked by a simple matching if the firm involved in the block is willing to form the block and there are sufficiently many workers of each type willing to join the block. The formal definition is as below.

Definition 6. A matching μ is *blocked* by a simple matching (f, x) if $x \succ_f \mu_f$ and

$$m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta) + \sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'})} \mu_{f'}(\theta) \geq x(\theta)$$

for all worker types $\theta \in \Theta$. The *core* of the labor market is the set of matchings that are not blocked by any simple matching.

In the definition, $m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta)$ on the left-hand side of the inequality is the mass of unemployed type θ workers and $\sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'})} \mu_{f'}(\theta)$ is the mass of type θ workers who are employed but are willing to join the block (f, x). The right-hand side of the inequality is the mass of type θ workers necessary to form the block. In principle, we may consider the notion of blocking by any nontrivial matching. However, it is without loss to focus on blocking by simple matchings in this setting because each agent can participate in no more than one contract at a time. Note that nonsimple matchings are combinations of simple matchings. If a nonsimple matching blocks some other matching, each of its simple components must also block it.

To apply our main theorem (Theorem 1), let us consider the following matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i \in I}, \sqsupset\}$ induced by the labor market. The set I of players is the set of firms and worker types, i.e., $I = F \cup \Theta$. The set $\mathcal{M}$ corresponds to the set of individually rational matchings. For each matching $\mu \in \mathcal{M}$, let the participation value $\phi_f(\mu)$ of firm be 0 if $\mu_f = \mathbf{0}$ and 1 otherwise, and let the participation value $\phi_\theta(\mu)$ of worker type θ be the fraction of employed type θ workers, i.e., $\phi_\theta(\mu) := \sum_{f \in F} \mu_f(\theta)/m(\theta)$. For each firm f, the relation $\trianglerighteq_f$ simply corresponds to firm f's preferences extended to $\mathcal{M}_f := \{\mu \in \mathcal{M} : \mu_f \neq \mathbf{0}\}$, i.e., we let $\mu' \trianglerighteq_f \mu$ if $\mu'_f \succsim_f \mu_f$. For each worker type θ, the aggregate preference relation $\trianglerighteq_\theta$ on $\mathcal{M}_\theta := \{\mu \in \mathcal{M} : \sum_{f \in F} \mu_f(\theta) > 0\}$ is defined as follows. First, we let $(f, x) \triangleright_\theta \mu$ whenever μ is nonsimple.[14] Second, for two simple matchings (f', x') and (f, x), let $(f', x') \trianglerighteq_\theta (f, x)$ if $(f', x') \succsim_\theta (f, x)$. For the relation $\sqsupset$, we let $\hat{\mu} \not\sqsupset \mu$ whenever

[14] We defined $\trianglerighteq_\theta$ in this way for convenience because only the relation $\trianglerighteq_\theta$ restricted to simple matchings matters for the analysis. Alternatively, we could define $\trianglerighteq_\theta$ by comparing the worst type θ workers under two matchings in general, but this will not make a difference.

$\hat{\mu}$ is nonsimple. For a simple matching (f, x), we let $(f, x) \sqsupset \mu$ if $x \succ_f \mu_f$ and

$$m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta) + \sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'})} \mu_{f'}(\theta) > 0$$

for all worker types $\theta \in \Theta$ with $x(\theta) > 0$. Note that the relation $\sqsupset$ defined above is less demanding than blocking in Definition 6, where the right-hand side of the inequality is $x(\theta)$ instead of 0. Therefore, the core of the matching game is even stronger than the core of the labor market.

Proposition 3. *With convex and upper semi-continuous preferences and within-type competition, the matching game induced by the large-firm labor market with peer preferences is regular and convex. Therefore, the core of the labor market is nonempty by Theorem 1.*

Proof. See Appendix A. □

The assumption of within-type competition is indispensable to the nonempty core result. We may adapt Example 4 in the Introduction to show that the core may be empty when the assumption fails. Consider a labor market with two firms and three types of workers, A, B, and C, each of which is of mass 1. The two firms regard all three types of workers as perfect substitutes and wish to hire as many workers as possible. All workers consider the two firms perfect substitutes and are only concerned about their colleagues. If a type A worker works with a set of colleagues of type distribution (x_A, x_B, x_C), her utility is

$$u_A(x_A, x_B, x_C) = (100 + 2\sqrt{x_B + 1} + \sqrt{x_C + 1}) \cdot \sqrt{x_A + 1} - 10000 \cdot \max\{0, x_A + x_B + x_C - 2\}.$$

If unemployed, her utility is 0. The utility function of the other two types of workers are cyclic symmetric to type A's. We can show that the core of this labor market is empty despite that the firms and workers have convex and continuous preferences. To see the empty core, note that there is strong within-type synergy to the extent that it is always beneficial for all workers of the same type to join one firm. Additionally, due to the last term of the utility function, the size of a firm will never exceed 2. Therefore, this labor market is essentially equivalent to the three-individual roommate example (Example 1), which we know has an empty core.

5 Stability

This section provides a unified understanding of the stability results of Azevedo and Hatfield (2018) and Che, Kim, and Kojima (2019) in the special case of labor markets with a continuum of workers of finitely many types. Azevedo and Hatfield (2018) consider a small-firm setting in which there is a continuum of firms and a continuum of workers and each firm employs finitely many workers. By contrast, Che, Kim, and Kojima (2019) consider a large-firm setting with finitely many firms, each of which is matched to a continuum of workers. We show that these two different settings with stability as the solution concept both satisfy the notion of convex matching games and therefore that stable matchings exist despite the difficulty of complementary preferences that typically leads to the nonexistence of stable matchings in a finite market.

The result in Section 4.1 implies a nonempty core in the small-firm setting of Azevedo and Hatfield (2018), and the result in Section 4.3 implies a nonempty core in the large-firm setting of Che, Kim, and Kojima (2019). However, in many-to-one matching models, the notion of stability is stronger than that of the core, and so the nonempty core results do not imply the existence of stable matchings. Under the notion of the core, to form a block, a firm cannot retain any worker it has already employed since such a worker does not strictly benefit from the block. By contrast, under the notion of stability, to form a block, a firm may retain some of its employees while hiring some new workers. This difference makes it easier to form a block under the notion of stability and therefore makes the stability notion stronger than that of the core.[15]

Fortunately, the framework of convex matching games can also be used to obtain stability results. Because our main theorem (Theorem 1) asserts the existence of matchings that are not blocked according to the blocking relation $\sqsupset$, if $\sqsupset$ corresponds to the notion of blocking under stability, the core of the matching game, as defined in Definition 1, will become the set of stable matchings. To maintain the applicability of the main theorem, we will modify the aggregate preference relation $\trianglerighteq_i$ s.t. the induced matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i \in I}, \sqsupset\}$ still satisfies the notion of convex matching games.

[15]This relation only holds in many-to-one matching problems. In more general settings such as many-to-many matching problems, the two notions are not related in a straightforward way.

5.1 Small-firm Labor Market

In this subsection, let us consider the small-firm setting of Azevedo and Hatfield (2018). Let $I = F \cup \Theta$ be the finite set of agent types, where F is the set of types of firms and Θ is the set of types of workers. Let $m(f) > 0$ be the mass of type f firms and $m(\theta) > 0$ be the mass of type θ workers. We assume that for some integer N each firm can hire no more than N workers.

Let us regard the employment relation between a firm and all its employees as one contract. Therefore, a contract type (f, x) is specified by the type f of the firm and quantity $x(\theta)$ of type θ workers θ employed by the firm. Let $X := \{x \in \mathbb{Z}_+^\Theta : \sum_{\theta\in\Theta} x(\theta) \leq N\}$ be the set of type distributions of employees. Let $X_+ := X\backslash\{\mathbf{0}\}$ be the set of nonzero distributions, so the set of all contract types is $F \times X_+$. A matching μ is specified by the type distribution of contracts, or formally, it is a measure on $F \times X_+$.[16] Feasibility requires that $\sum_{x\in X_+} \mu(f, x) \leq m(f)$ for each firm type f and $\sum_{(f,x)\in F\times X_+} \mu(f, x) \cdot x(\theta) \leq m(\theta)$ for each worker type θ. Clearly, the difference $m(f) - \sum_{x\in X_+} \mu(f, x) =: \mu(f, \mathbf{0})$ is the mass of type f firms employing no workers and the difference $m(\theta) - \sum_{(f,x)\in F\times X_+} \mu(f, x) \cdot x(\theta)$ is the mass of unemployed type θ workers.

Type f firms have a complete and transitive preference relation $\succsim_f$ on X and type θ workers have a complete and transitive preference relation $\succsim_\theta$ on $F \cup \{\emptyset\}$, where $\emptyset$ represents the state of being unemployed. We say that a contract type $(f, x) \in F \times X_+$ is individually rational if $x \succsim_f \mathbf{0}$ and $f \succsim_\theta \emptyset$ for all θ with $x(\theta) > 0$. A matching μ is individually rational if its support $\{(f, x) \in F \times X_+ : \mu(f, x)\} > 0\}$ only contains individually rational contract types. In the following discussion, a matching always refers to an individually rational matching.

To define stability, let us use the term "s-block" to distinguish it from "c-block", the notion of blocking under the core as in Definition 4. We say that a matching μ is *simple* if its support is a singleton. Because in our current setting, each agent can participate in no more than one contract at a time, it is without loss to focus on blocking by a simple matching. Note that nonsimple matchings are combinations of simple matchings and that if a nonsimple matching blocks some other matching, each of its simple components must

[16] Note that we only need to keep track of the measure on contract type (f, x)s (μ_x in Section 4.1) since the measure on bundles of roles (μ_i in Section 4.1) is redundant. In this model, a typical role for type f firms is "being the firm in a contract of type (f, x)" and a typical role for type θ workers is "being a type θ worker in a contract of type (f, x)", and therefore, the roles are uniquely pinned down by the associated agent type and contract type. Then, the measure on bundles is also pinned down because each agent can play no more than one role at a time.

also block it.

Definition 7. A matching μ is *s-blocked* by a simple matching with support $\{(f, x)\}$ if there exists $(f, x') \in F \times X$ with $\mu(f, x') > 0$ s.t. (1) $x \succ_f x'$ if $x' \neq \mathbf{0}$ and (2) for each worker type θ with $x(\theta) > 0$, either we have $x'(\theta) > 0$, or $\sum_{(f,x) \in F \times X_+} \mu(f, x) \cdot x(\theta) < m(\theta)$, or there exists (f'', x'') in the support of μ with $x''(\theta) > 0$ and $f'' \prec_\theta f$. A matching in $\mathcal{M}$ is *stable* if it is not s-blocked by any simple matching.

Condition (2) in the definition above requires that for each type θ of workers necessary to form the s-block, either the firm participating in the s-block has already employed some type θ workers, or some type θ workers are unemployed, or some type θ workers strictly prefer the firm to their current employer. By contrast, the notion of the c-block (Definition 4) does not regard the first group of workers, i.e., those already employed by the firm, as available participants in a block since those workers do not strictly benefit from the block.[17] Therefore, it is easier to form an s-block than to form a c-block, and so stability is stronger than the notion of the core in this setting.

To apply our main theorem (Theorem 1), let us consider the following matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i \in I}, \sqsupset\}$ induced by the labor market. Let I, the set of players, be the set of firms and worker types, i.e., $I = F \cup \Theta$. Let $\mathcal{M}$ be the set of individually rational matchings. For each matching $\mu \in \mathcal{M}$, let the participation value $\phi_f(\mu)$ of firm type f be the fraction of type f firms employing some workers, i.e., $\phi_f(\mu) = \sum_{x \in X_+} \mu(f, x)/m(f)$, and let the participation value $\phi_\theta(\mu)$ of worker type θ be the fraction of employed type θ workers, i.e., $\phi_\theta(\mu) := \sum_{(f,x) \in F \times X_+} \mu(f, x) \cdot x(\theta)/m(\theta)$.

For each agent type $i \in F \cup \Theta$, let the relation $\trianglerighteq_i$ be defined s.t. $\mu' \triangleright_i \mu$ if μ' is simple and μ is nonsimple. For two simple matchings $\mu, \mu' \in \mathcal{M}_f$, let their support be $\{(f, x)\}$ and $\{(f, x')\}$, respectively, and the relation $\trianglerighteq_f$ is defined s.t. $\mu' \trianglerighteq_f \mu$ if $x' \succsim_f x$. For two simple matchings $\mu, \mu' \in \mathcal{M}_\theta$, let their support be $\{(f, x)\}$ and $\{(f', x')\}$, respectively, and the relation $\trianglerighteq_\theta$ is defined as follows. First, we arbitrarily break ties for $\succsim_\theta$ to obtain a strict preference ordering $\succsim'_\theta$ on F and let $\mu' \triangleright_\theta \mu$ if $f' \succ'_\theta f$. Second, if $f' = f$, we let $\mu' \trianglerighteq_\theta \mu$ if $x' \succsim_f x$, i.e., we use the firm's preferences as a tie-breaker in the second step.

For the blocking relation $\sqsupset$, we let $\hat{\mu} \not\sqsupset \mu$ if $\hat{\mu}$ is nonsimple. If $\hat{\mu}$ is simple, we let $\hat{\mu} \sqsupset \mu$ if $\hat{\mu}$ s-blocks μ according to Definition 7.

[17] Formally, by applying Definition 4 to the current setting, a matching μ is c-blocked by a simple matching with the support $\{(f, x)\}$ if there exists $(f, x') \in F \times X$ with $\mu(f, x') > 0$ s.t. (1) $x \succ_f x'$ if $x' \neq \mathbf{0}$ and (2) for each worker type θ with $x(\theta) > 0$, either we have $\sum_{(f,x) \in F \times X_+} \mu(f, x) \cdot x(\theta) < m(\theta)$ or there exists (f'', x'') in the support of μ with $x''(\theta) > 0$ and $f'' \prec_\theta f$. The difference is that it does not accept the first case $x'(\theta) > 0$ as in the notion of the s-block.

Proposition 4 (Azevedo and Hatfield (2018))**.** *Without peer preferences, the matching game induced by the small-firm labor market model with stability as its solution concept is regular and convex. Therefore, stable matchings exist by Theorem 1.*

Proof. See Appendix A. □

Note that the original stability result in Azevedo and Hatfield (2019) is more general than the proposition above. Their result also applies to many-to-many matching where agents on one side of the market have substitutable preferences, which subsumes many-to-one matching as a special case.

5.2 Large-firm Labor Market

In this subsection, let us consider the large-firm setting of Che, Kim, and Kojima (2019). As in Section 4.3, let F be the finite set of firms and Θ be the finite set of worker types. For each worker type $\theta \in \Theta$, let $m(\theta) > 0$ be the mass of type θ workers. Let $X := \prod_{\theta\in\Theta}[0, m(\theta)]$ be the set of all worker type distributions. A matching μ specifies the type distribution of the set of employees of each firm $f \in F$. Formally, a matching is defined as $\mu = \{\mu_f\}_{f\in F}$, where $\mu_f \in X$ represents the type distributions of the employees of firm f. Feasibility requires that $\sum_{f\in F} \mu_f \leq m$, and the difference $m - \sum_{f\in F} \mu_f$ is the type distribution of unemployed workers. Each firm f has a complete and transitive preference relation $\succsim_f$ on X, which is assumed to be convex and continuous. Each type θ worker has a complete and transitive preference relation $\succsim_\theta$ on $F \cup \{\emptyset\}$, where the symbol $\emptyset$ represents the state of being unemployed. This assumes away workers' peer preferences.

We say that a matching $\mu = \{\mu_f\}_{f\in F}$ is individually rational if $\mu_f \succsim_f \mathbf{0}$ for each firm f and $f \succsim_\theta \emptyset$ for each worker type θ with $\mu_f(\theta) > 0$. Let $\mathcal{M}$ be the set of all individually rational matchings. In the following discussion, a matching always refers to an individually rational matching.

To define stability, let us use the term "s-block" to distinguish it from "c-block", the notion of blocking under the core in Definition 6. A *simple* matching (f, x) refers to a matching μ with $\mu_f = x \neq \mathbf{0}$ and $\mu_{f'} = \mathbf{0}$ for all $f' \neq f$. Again, it is without loss to focus on blocking by simple matchings.

Definition 8. A matching μ is *s-blocked* by a simple matching (f, x) if $x \succ_f \mu_f$ and

$$\mu_f(\theta) + m(\theta) - \sum_{f'\in F} \mu_{f'}(\theta) + \sum_{f'\in F: f' \prec_\theta f} \mu_{f'}(\theta) \geq x(\theta)$$

for all worker types $\theta \in \Theta$. A matching in $\mathcal{M}$ is *stable* if it is not s-blocked by any simple matching.

The left-hand side of the inequality in the definition is the mass of type i agents available for block (f, x), and the right-hand side is the mass of type θ workers necessary to form the block. Note that on the left-hand side the term $\mu_f(\theta)$ is the mass of type θ workers who are already employed by firm f, $m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta)$ is the mass of unemployed type θ workers, and $\sum_{f' \in F: f' \prec_\theta f} \mu_{f'}(\theta)$ is the mass of type θ workers who are employed by other some other firm f' strictly less preferred to f. By contrast, the notion of the c-block (Definition 6) does not regard the first group of workers, i.e., those already employed by the firm, as available participants in a block since those workers do not strictly benefit from the block.[18] Therefore, it is easier to form an s-block than to form a c-block, and so stability is stronger than the notion of the core in this setting.

To apply our main theorem (Theorem 1), let us consider the following matching game $\{I, \mathcal{M}, \phi, (\trianglerighteq_i)_{i \in I}, \sqsupset\}$ induced by the labor market. Let I, the set of players, be the set of firms and worker types, i.e., $I = F \cup \Theta$. Let $\mathcal{M}$ be the set of individually rational matchings. For each matching $\mu \in \mathcal{M}$, let the participation value $\phi_f(\mu)$ of firm f be 0 if $\mu_f = \mathbf{0}$ and be 1 otherwise, and let the participation value $\phi_\theta(\mu)$ of worker type θ be the fraction of employed type θ workers, i.e., $\phi_\theta(\mu) := \sum_{f \in F} \mu_f(\theta)/m(\theta)$.

For each firm f, the relation $\trianglerighteq_f$ simply corresponds to firm f's preferences extended to $\mathcal{M}_f := \{\mu \in \mathcal{M} : \mu_f \neq \mathbf{0}\}$, i.e., we let $\mu' \trianglerighteq_f \mu$ if $\mu'_f \succsim_f \mu_f$. For each worker type θ, the aggregate preference relation $\trianglerighteq_\theta$ on $\mathcal{M}_\theta := \{\mu \in \mathcal{M} : \sum_{f \in F} \mu_f(\theta) > 0\}$ is defined as follows. First, we let $(f, x) \triangleright_\theta \mu$ whenever μ is nonsimple. Second, we arbitrarily break ties for $\succsim_\theta$ to obtain a strict preference ordering $\succsim'_\theta$ on F, and for two simple matchings (f', x') and (f, x) with $f' \neq f$, let $(f', x') \triangleright_\theta (f, x)$ if $f' \succ'_\theta f$. Finally, for two simple matchings (f, x') and (f, x) associated with the same firm f, we let $(f, x') \trianglerighteq_\theta (f, x)$ if $x'(\theta) \leq x(\theta)$.

For the blocking relation $\sqsupset$, we let $\hat{\mu} \not\sqsupset \mu$ whenever $\hat{\mu}$ is nonsimple. For a simple

[18] More formally, by applying Definition 6 to our current setting, a matching μ is c-blocked by a simple matching (f, x) if $x \succ_f \mu_f$ and

$$m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta) + \sum_{f': f' \prec_\theta f} \mu_{f'}(\theta) \geq x(\theta).$$

The difference is that $\mu_f(\theta)$ is not counted towards the left-hand side as in the notion of the s-block.

matching (f, x), we let $(f, x) \sqsupset \mu$ if $x \succ_f \mu_f$ and

$$\mu_f(\theta) + m(\theta) - \sum_{f' \in F} \mu_{f'}(\theta) + \sum_{f' \in F : f' \prec_\theta f} \mu_{f'}(\theta) > x(\theta)$$

for all worker types $\theta \in \Theta$ with $x(\theta) > 0$. Note that the blocking relation $\sqsupset$ here is slightly more demanding than s-blocking in Definition 8 because the inequality above is strict.[19] Despite this slight difference, the set of unblocked matchings in the sense of $\sqsupset$ is exactly the set of stable matchings in this labor market because if a matching μ is s-blocked by a simple matching (f, x), there exists $x' < x$ close enough to x s.t. $(f, x') \sqsupset \mu$ due to the continuity of $\succsim_f$.

Proposition 5 (Che, Kim, and Kojima (2019))**.** *If firms' preferences are continuous and convex and workers have no peer preferences, the matching game induced by the large-firm labor market model with stability as its solution concept is regular and convex. Therefore, stable matchings exist by Theorem 1.*

Proof. See Appendix A. □

Although our setting focuses on a finite set of worker types, Proposition 5 can be generalized to allow for a compact set of worker types as in Che, Kim, and Kojima (2019) using the standard argument that a compact set can be approximated arbitrarily well by a finite set. See Appendix B for details.

6 Conclusion

This paper provides a nonempty core result in a class of games we label "convex matching games", which may allow for arbitrary contracting networks, multilateral contracts, and complementary preferences. Using Scarf's lemma, we show that the core is nonempty in all regular and convex matching games.

The structure of the core of matching games in general remains an open question. Azevedo and Leshno (2016) find that the stable matching is unique in a large-firm labor market when firms have responsive preferences. Later, this uniqueness result is generalized by Che, Kim, and Kojima (2019) to preferences that satisfy substitutability, the law of

[19] Here, we cannot define $\sqsupset$ exactly as s-blocking because doing so will may violate the closedness of $NB(\mu)$, and therefore, we cannot invoke Theorem 1.

aggregate demand, and a richness condition. However, I find that it is not straightforward to further generalize this uniqueness result to matching games in general. In fact, the core of a convex matching game may even fail to be convex. Consider, for example, the following continuum roommate problem with four types of individuals, A, B, C, and D, each of which is of mass 1. Assume that type A individuals have high income, type D individuals have low income, and type B and type C individuals have middle income. Each individual prefers a roommate with higher income, but for some other reason, they consider matching with another individual of the same type unacceptable. Consider the matching μ^1 in which all type A individuals are matched with type B individuals and all type C individuals are matched with type D individuals, and we can verify that it is in the core. By symmetry, the matching μ^2 in which all type A individuals are matched with type C individuals and all type B individuals are matched with type D individuals is also in the core. However, note that any convex combination of μ^1 and μ^2 is not in the core since it is blocked by type B and type C individuals who are matched to a type D individual. Therefore, the core, which is equivalent to the set of stable matchings in this one-to-one matching setting, is not even a convex set.

Appendix A: Proofs

Proof of Proposition 1. **First, we show that the matching game is regular.**

(i) $\mathcal{M}$ is compact.

For a compact metric space S and a nonnegative real number c, let $BM(S, c)$ be the set of all Borel measures on S with the measure of S no greater than c. By Banach-Alaoglu theorem, we know that $BM(S, c)$ is compact w.r.t. the weak-* topology.[20]

Recall that the set of individually rational bundles for type i agents is

$$\mathfrak{B}_i = \{\beta_i \in BM(R_i, N) : \beta_i \text{ is integer-valued, } \beta_i \neq \mathbf{0}_i, \text{ and } \beta_i \succsim_i \mathbf{0}_i\}.$$

We claim that $\mathfrak{B}_i$ is closed in $BM(R_i, N)$ and therefore also compact. To see this, take a sequence of bundles (β_i^k) in $\mathfrak{B}_i$ convergent to $\beta_i^0 \in BM(R_i, N)$. It is sufficient to show that $\beta_i^0 \in \mathfrak{B}_i$.[21] First, to see that β_i^0 is integer-valued, suppose that $\beta_i^0(\{r_i\})$ is not an integer for some role $r_i \in R_i$. Let $a < \beta_i^0(\{r_i\}) < a + 1$ where a is an integer. Then there

[20]See, for example, Lemma 3 in Appendix A of Che, Kim, and Kojima (2019) for more detailed arguments.

[21]Note that in $BM(R_i, N)$, closedness is equivalent to sequential closedness because the weak-* topology is metrizable by, for example, the Prokhorov metric introduced by Prokhorov (1956).

exists a closed ball $\bar{B}$ centered at r_i with positive radius s.t. $\beta_i^0(\bar{B}) < a+1$. Consider the interior of $\bar{B}$, i.e., the open ball B, and we have $\beta_i^0(B) \geq \beta_i^0(\{r_i\}) > a$. By Portmanteau theorem[22], we have $\limsup \beta_i^k(\bar{B}) \leq \beta_i^0(\bar{B})$ and $\liminf \beta_i^k(B) \geq \beta_i^0(B)$. Because each β_i^k is integer-valued, for sufficiently large k, we have $\beta_i^k(\bar{B}) \leq a$ and $\beta_i^k(B) \geq a+1$, which contradicts $B \subset \bar{B}$. Second, to see $\beta_i^0 \neq \mathbf{0}_i$, note that $\beta_i^k(R_i) \geq 1$ and $\beta_i^k(R_i) \to \beta_i^0(R_i)$ implies $\beta_i^0(R_i) \geq 1$. Third, $\beta_i^k \succsim_i \mathbf{0}_i$ for each k implies $\beta_i^0 \succsim_i \mathbf{0}_i$ in the limit because $\succsim_i$ is continuous. Therefore, $\mathfrak{B}_i$ is compact, which in turn implies that the space $BM(\mathfrak{B}_i, m_i)$ where μ_i lies is also compact.

Because X does not contain the empty contract, we have $\sum_{i \in I} x_i(R_i) > 0$ for all $x \in X$. Because $x_i(R_i)$ is continuous in x and X is compact, we know that $\sum_{i \in I} x_i(R_i) \geq \delta$ for some $\delta > 0$. So for each matching μ, we have

$$\int_X \sum_{i \in I} x_i(R_i) d\mu_x \geq \int_X \delta d\mu_x \geq \delta \cdot \mu_x(X),$$

which implies

$$\begin{aligned} \mu_x(X) &\leq \delta^{-1} \sum_{i \in I} \int_X x_i(R_i) d\mu_x = \delta^{-1} \sum_{i \in I} \int_{\mathfrak{B}_i} \beta_i(R_i) d\mu_i \leq \delta^{-1} \sum_{i \in I} \int_{\mathfrak{B}_i} N d\mu_i \\ &= \delta^{-1} N \sum_{i \in I} \mu_i(\mathfrak{B}_i) \leq \delta^{-1} N \sum_{i \in I} m_i. \end{aligned} \tag{2}$$

Therefore the measure μ_x lies in the compact space $BM(X, \delta^{-1} N \sum_{i \in I} m_i)$.

Recall that the set of individually rational matchings is

$$\mathcal{M} = \{(\mu_x, (\mu_i)_{i \in I}) \in \bar{\mathcal{M}} : \int_{\mathfrak{B}_i} \beta_i d\mu_i = \int_X x_i d\mu_x \text{ for each } i\},$$

where $\bar{\mathcal{M}} := BM(X, \delta^{-1} N \sum_{i \in I} m_i) \times \prod_{i \in I} BM(\mathfrak{B}_i, m_i)$. By Tychonoff theorem, the product space $\bar{\mathcal{M}}$ is compact since we have shown that each of its component is compact. To show compactness of $\mathcal{M}$, it is sufficient to show that $\mathcal{M}$ is closed in $\bar{\mathcal{M}}$. Take any sequence (μ^k) in $\mathcal{M}$ convergent to some $\mu^0 \in \bar{\mathcal{M}}$. We want to show $\int_{\mathfrak{B}_i} \beta_i d\mu_i^0 = \int_X x_i d\mu_x^0$, i.e., $\int_{\mathfrak{B}_i} \beta_i d\mu_i^0$ and $\int_X x_i d\mu_x^0$ are the same Borel measure on R_i. Note that the sequence of measures $\int_{\mathfrak{B}_i} \beta_i d\mu_i^k$ converges to $\int_{\mathfrak{B}_i} \beta_i d\mu_i^0$ and the the sequence of measures $\int_X x_i d\mu_x^k$ con-

[22] See, for example, Theorem 2.8.1 of Ash and Doleans-Dade (2009).

verges to $\int_X x_i d\mu_x^0$ in weak-* topology.[23] Because $\mu^k \in \mathcal{M}$ implies $\int_{\mathfrak{B}_i} \beta_i d\mu_i^k = \int_X x_i d\mu_x^k$ for each k, we have $\int_{\mathfrak{B}_i} \beta_i d\mu_i^0 = \int_X x_i d\mu_x^0$ in the limit. Therefore, the set $\mathcal{M}$ is compact.

(ii) $NB(\mu)$ is closed for each nontrivial matching μ.

By Definition 4, we have

$$\begin{aligned} NB(\mu) &= \{\mu' \in \mathcal{M} : \mu \not\sqsupset \mu'\} \\ &= \bigcup_{i:\mu_i(\mathfrak{B}_i)>0} \{\mu' \in \mathcal{M} : \mu_i'(\mathfrak{B}_i) = m_i \text{ and } \mu_i'(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) = 0\}. \end{aligned}$$

To show closedness of $NB(\mu)$, it is sufficient to show that the set

$$NB_i(\mu) := \{\mu' \in \mathcal{M} : \mu_i'(\mathfrak{B}_i) = m_i \text{ and } \mu_i'(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) = 0\}$$

is closed for each i with $\mu_i(\mathfrak{B}_i) > 0$. Take any sequence (μ^k) in $NB_i(\mu)$ convergent to some $\mu^0 \in \mathcal{M}$. By Portmanteau theorem, we have $\mu_i^0(\mathfrak{B}_i) = \lim \mu_i^k(\mathfrak{B}_i) = m_i$ and

$$\mu_i^0(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) \leq \liminf \mu_i^k(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) = 0$$

because the set $\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}$ is open by continuity of $\succsim_i$. Therefore we have $\mu^0 \in NB_i(\mu)$ and so $NB(\mu)$ is closed.

Second, we show that the matching game is convex.

(i) Let $\mu^* = \sum_{j=1}^n w^j \mu^j$ be a ϕ-convex combination of finitely many matchings $\{\mu^j\}_{j=1}^n$ in $\mathcal{M}$. We need to show that $\mu^* \in \mathcal{M}$.

To do so, we verify that

$$\mu_i^*(\mathfrak{B}_i) = \sum_{j=1}^n w^j \mu_i^j(\mathfrak{B}_i) = m_i \sum_{j=1}^n w^j \phi_i(\mu^j) \leq m_i,$$

which implies $\mu_i^* \in BM(\mathfrak{B}_i, m_i)$. Besides, the accounting identity also holds under ϕ-

[23] To see this, note that for any continuous function f_i on R_i, we have

$$\int_{\mathfrak{B}_i} (\int_{R_i} f_i d\beta_i) d\mu_i^k \to \int_{\mathfrak{B}_i} (\int_{R_i} f_i d\beta_i) d\mu_i^0$$

and

$$\int_X (\int_{R_i} f_i dx_i) d\mu_x^k \to \int_X (\int_{R_i} f_i dx_i) d\mu_x^0$$

because the continuity of f_i implies that $\int_{R_i} f_i d\beta_i$ and $\int_{R_i} f_i dx_i$ are continuous functions on $\mathfrak{B}_i$ and X respectively.

convex combinations since

$$\int_{\mathfrak{B}_i} \beta_i d\mu_i^* = \sum_{j=1}^n w^j \int_{\mathfrak{B}_i} \beta_i d\mu_i^j = \sum_{j=1}^n w^j \int_X x_i d\mu_x^j = \int_X x_i d\mu_x^*.$$

Following the same arguments in Inequality (2) we have $\mu_x^*(X) \leq \delta^{-1} N \sum_{i\in I} m_i$, which implies $\mu_x^* \in BM(X, \delta^{-1} N \sum_{i\in I} m_i)$. Therefore, we know that $\mu^* \in \mathcal{M}$.

(ii) If $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ s.t. $\phi_i(\mu) > 0$, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, we need to show that $\mu \not\trianglerighteq \mu^*$.

For the agent i found by the existence statement above, we have

$$\mu_i^*(\mathfrak{B}_i) = \sum_{j=1}^n w^j \mu_i^j(\mathfrak{B}_i) = m_i \sum_{j=1}^n w^j \phi_i(\mu^j) = m_i,$$

i.e., all type i agents are matched under μ^*. Moreover, for each j with $w^j > 0$ and $\mu_i^j(\mathfrak{B}_i) > 0$, we have $\phi_i(\mu^j) > 0$ and so $\mu^j \trianglerighteq_i \mu$, i.e., $\beta_i \succsim_i \underline{\beta}_i(\mu)$ for all β_i in the support of μ_i^j. This implies[24] $\mu_i^j(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) = 0$, and therefore

$$\mu_i^*(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\}) = \sum_{j: w^j > 0, \mu_i^j(\mathfrak{B}_i) > 0} w^j \mu_i^j(\{\beta_i \in \mathfrak{B}_i : \beta_i \prec_i \underline{\beta}_i(\mu)\} = 0.$$

Therefore, we have shown that no type i agent is willing to accept the worst bundle under μ. Because $\phi_i(\mu) > 0$ implies $\mu_i(\mathfrak{B}_i) > 0$, i.e., the participation of type i agent is necessary to form the block μ, we have $\mu \not\trianglerighteq \mu^*$ by Definition 4. This completes the proof of convexity of the matching game. $\square$

Proof of Proposition 2. **First, we show that the matching game is regular.**

(i) $\mathcal{M}$ is compact.

Recall that the set $\mathcal{M}$ of individually rational matchings is

$$\mathcal{M} = \{\mu \in \prod_{\omega\in\Omega} \mathcal{M}_\omega : \mu \succsim_i \mathbf{0} \text{ for each } i\},$$

[24]This implication relies on $\mathfrak{B}_i$ admitting a countable basis, which is guaranteed here by compactness and metrizability of $\mathfrak{B}_i$.

which is compact because each $\mathcal{M}_\omega$ is compact and $\succsim_i$ is upper semi-continuous.

(ii) $NB(\mu)$ is closed for each nontrivial matching μ.

By Definition 5, we have

$$NB(\mu) = \{\mu' \in \mathcal{M} : \mu \not\sqsupset \mu'\} = \bigcup_{i \in a(\mu)} \{\mu' \in \mathcal{M} : \mu' \succsim_i \mu\},$$

which is a closed set because each $\{\mu' \in \mathcal{M} : \mu' \succsim_i \mu\}$ is closed by upper semi-continuity of $\succsim_i$.

Second, we show that the matching game is convex.

(i) Let $\mu^* = \sum_{j=1}^n w^j \mu^j$ be a ϕ-convex combination of finitely many matchings $\{\mu^j\}_{j=1}^n$ in $\mathcal{M}$. We need to show that $\mu^* \in \mathcal{M}$.

For each venture $\omega \in \Omega$, let i be an arbitrary agent in $a(\omega)$ and we have

$$\mu^*_\omega = \sum_{j=1}^n w^j \mu^j_\omega = \sum_{j: w^j > 0, i \in a(\mu^j)} w^j \mu^j_\omega + (1 - \sum_{j: w^j > 0, i \in a(\mu^j)} w^j) \cdot \mathbf{0}_\omega. \tag{3}$$

This shows that μ^*_ω is a convex combination of μ^j_ω's and $\mathbf{0}_\omega$ since $\sum_{j: w^j > 0, i \in a(\mu^j)} w^j = \sum_{j=1}^n w^j \phi_i(\mu^j) \leq 1$. Because μ^j_ω's and $\mathbf{0}_\omega$ are in $\mathcal{M}_\omega$, we have $\mu^*_\omega \in \mathcal{M}_\omega$ by convexity of $\mathcal{M}_\omega$. Therefore we have $\mu^* \in \prod_{\omega \in \Omega} \mathcal{M}_\omega$.

Moreover, for each agent i, Equation (3) holds for all $\omega \in \Omega_i$ and the weights of the convex combination do not depend on ω. This implies that $\{\mu^*_\omega\}_{\omega \in \Omega_i}$ is a convex combination of $\{\mu^j_\omega\}_{\omega \in \Omega_i}$'s and $\{\mathbf{0}_\omega\}_{\omega \in \Omega_i}$. Because $\mu^j \in \mathcal{M}$ implies $\{\mu^j_\omega\}_{\omega \in \Omega_i} \succsim_i \{\mathbf{0}_\omega\}_{\omega \in \Omega_i}$ for each j, by convexity of $\succsim_i$, we have $\{\mu^*_\omega\}_{\omega \in \Omega_i} \succsim_i \{\mathbf{0}_\omega\}_{\omega \in \Omega_i}$, i.e., $\mu^* \succsim_i \mathbf{0}$, which implies $\mu^* \in \mathcal{M}$.

(ii) If $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ s.t. $\phi_i(\mu) > 0$, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, we need to show that $\mu \not\sqsupset \mu^*$.

For the agent i found by the existence statement above, we have $\sum_{j: w^j > 0, i \in a(\mu^j)} w^j = \sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and therefore $\mu^*_\omega = \sum_{j: w^j > 0, i \in a(\mu^j)} w^j \mu^j_\omega$ implies that μ^*_ω is a convex combination of those μ^j_ω's. For each j with $w^j > 0$ and $i \in a(\mu^j)$, we have $\phi_i(\mu^j) = 1 > 0$ and therefore $\mu^j \trianglerighteq_i \mu$, which implies $\{\mu^j_\omega\}_{\omega \in \Omega_i} \succsim_i \{\mu_\omega\}_{\omega \in \Omega_i}$. By convexity of $\succsim_i$, we have $\{\mu^*_\omega\}_{\omega \in \Omega_i} \succsim_i \{\mu_\omega\}_{\omega \in \Omega_i}$, i.e., $\mu^* \succsim_i \mu$. Therefore we have $\mu \not\sqsupset \mu^*$ because $\phi_i(\mu) > 0$ implies $i \in a(\mu)$. This completes the proof of convexity of the matching game. $\square$

Proof of Proposition 3. **First, we show that the matching game is regular.**

(i) $\mathcal{M}$ is compact.

Recall that the set $\mathcal{M}$ of individually rational matchings is

$$\mathcal{M} = \{\mu \in X^F : \sum_{f \in F} \mu_f \leq m, \mu_f \succsim_f \mathbf{0} \text{ for each } f, \text{ and } \mu_f(\theta) > 0 \text{ implies } (f, \mu_f) \succsim_\theta \emptyset\}.$$

To show closedness of $\mathcal{M}$, take a sequence of matchings (μ^k) in $NB(f,x)$ convergent to $\mu^0 \in X^F$, and we want to show $\mu^0 \in \mathcal{M}$. By upper semi-continuity of $\succsim_f$, $\mu_f^k \succsim_f \mathbf{0}$ implies $\mu_f^0 \succsim_f \mathbf{0}$ in the limit. Besides, $\mu_f^0(\theta) > 0$ implies $\mu_f^k(\theta) > 0$ for sufficiently large k, which in turn implies $(f, \mu_f^k) \succsim_\theta \emptyset$. Therefore we have $(f, \mu_f^0) \succsim_\theta \emptyset$ in the limit and so $\mu^0 \in \mathcal{M}$.

Because X^F is bounded, the set $\mathcal{M}$ is compact (w.r.t. the standard Euclidean topology).

(ii) $NB(\mu)$ is closed for each nontrivial matching μ.

If μ is nonsimple, the way we define the blocking relation $\sqsupset$ implies $NB(\mu) = \mathcal{M}$, which is a closed set. So we only need to consider the case in which μ is a simple matching. Let μ be (f, x).

Take a sequence of matchings (μ^k) in $NB(f,x)$ convergent to $\mu^0 \in \mathcal{M}$, and we want to show $\mu^0 \in NB(f,x)$. By definition of $\sqsupset$, $\mu^k \in NB(f,x)$ implies that either (a) $\mu_f^k \succsim_f x$ or (b)

$$m(\theta) - \sum_{f' \in F} \mu_{f'}^k(\theta) + \sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'}^k)} \mu_{f'}^k(\theta) = 0 \tag{4}$$

for some worker type $\theta \in \Theta$ with $x(\theta) > 0$. If there are infinitely many k's that satisfy condition (a) $\mu_f^k \succsim_f x$, by upper semi-continuity of $\succsim_f$ we have $\mu_f^0 \succsim_f x$ in the limit and so $\mu^0 \in NB(\mu)$. If there are only finitely many k's that satisfy condition (a), there must exist infinitely many k's that satisfy condition (b). By finiteness of Θ, there must exists some $\theta \in \Theta$ with $x(\theta) > 0$ s.t. Equality (4) holds along a subsequence of (μ^k). By upper semi-continuity of $\succsim_\theta$, $(f,x) \succ_\theta (f', \mu_{f'}^0)$ implies $(f,x) \succ_\theta (f', \mu_{f'}^k)$ for sufficiently large k, which in turn implies $\sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'}^0)} \mu_{f'}^k(\theta) \leq \sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'}^k)} \mu_{f'}^k(\theta)$ for sufficiently large k. Therefore, Equality (4) holds in the limit, i.e.,

$$m(\theta) - \sum_{f' \in F} \mu_{f'}^0(\theta) + \sum_{f' \in F : (f,x) \succ_\theta (f', \mu_{f'}^0)} \mu_{f'}^0(\theta) = 0,$$

and we have $\mu^0 \in NB(\mu)$. This completes the proof of regularity of the matching game.

Second, we show that the matching game is convex.

(i) Let $\mu^* = \sum_{j=1}^n w^j \mu^j$ be a ϕ-convex combination of finitely many matchings $\{\mu^j\}_{j=1}^n$ in $\mathcal{M}$. We need to show that $\mu^* \in \mathcal{M}$.

For each $\theta \in \Theta$, we have

$$\sum_{f\in F} \mu_f^*(\theta) = \sum_{f\in F}\sum_{j=1}^n w^j \mu_f^j(\theta) = \sum_{j=1}^n [w^j \sum_{f\in F} \mu_f^j(\theta)] = \sum_{j=1}^n [w^j \phi_\theta(\mu^j) m(\theta)] \le m(\theta).$$

Therefore, we have $\sum_{f\in F} \mu_f^* \le m$ and so μ^* is feasible. Next, to show $\mu_f^* \succsim_f \mathbf{0}$, note that $\sum_{j:\mu_f^j \neq \mathbf{0}} w^j = \sum_{j=1}^n w^j \phi_f(\mu^j) \le 1$, and so we have

$$\mu_f^* = \sum_{j=1}^n w^j \mu_f^j = \sum_{j:\mu_f^j\neq\mathbf{0}} w^j \mu_f^j + (1 - \sum_{j:\mu_f^j\neq\mathbf{0}} w^j)\cdot \mathbf{0},$$

i.e., μ_f^* is a convex combination of μ_f^j's and $\mathbf{0}$. Because $\mu^j \in \mathcal{M}$, we have $\mu_f^j \succsim_f \mathbf{0}$ and so $\mu_f^* \succsim_f \mathbf{0}$ by convexity of $\succsim_f$.

Moreover, we also need to show that $\mu_f^*(\theta) > 0$ implies $(f, \mu_f^*) \succsim_\theta \emptyset$. Suppose we have $\mu_f^*(\theta) > 0$. Note that

$$\mu_f^* = \sum_{j=1}^n w^j \mu_f^j = \sum_{j:\mu_f^j(\theta)>0} w^j \mu_f^j + \sum_{j:\mu_f^j\neq\mathbf{0},\mu_f^j(\theta)=0} w^j \mu_f^j + (1 - \sum_{j:\mu_f^j\neq\mathbf{0}} w^j)\cdot \mathbf{0},$$

i.e., μ_f^* is a convex combination of μ_f^j's and $\mathbf{0}$ since $\sum_{j:\mu_f^j\neq\mathbf{0}} w^j \le 1$. Because $\mu^j \in \mathcal{M}$, we have $(f, \mu_f^j) \succsim_\theta \emptyset$ for each j with $\mu_f^j(\theta) > 0$. Because $\sum_{j:\mu_f^j(\theta)>0} w^j \mu_f^j(\theta) = \mu_f^*(\theta) > 0$, there must exist at least one $\hat{j}$ with $\mu_f^{\hat{j}}(\theta) > 0$. By the assumption of within-type competition, we have $(f, \mathbf{0}) \succsim_\theta (f, \mu_f^{\hat{j}}) \succsim_\theta \emptyset$ and for each j with $\mu_f^j(\theta) = 0$, we have $(f, \mu_f^j) \succsim_\theta (f, \mu_f^{\hat{j}}) \succsim_\theta \emptyset$. By convexity of $\succsim_\theta$, we have $(f, \mu_f^*) \succsim_\theta \emptyset$. Therefore we have shown that $\mu^* \in \mathcal{M}$.

(ii) If $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ s.t. $\phi_i(\mu) > 0$, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, we need to show that $\mu \not\rhd \mu^*$.

This holds by definition if μ is nonsimple and so we only need to consider the case in which μ is a simple matching (f, x).

Case A: The player i is the firm f.

Note that for each j with $w^j > 0$ and $\mu_f^j \neq \mathbf{0}$, we have $\phi_f(\mu^j) = 1 > 0$ and so $\mu^j \trianglerighteq_f (f, x)$, which implies $\mu_f^j \succsim_f x$. Besides, note that $\sum_{j: w_j > 0, \mu_f^j \neq \mathbf{0}} w^j = \sum_{j=1}^n w^j \phi_f(\mu^j) = 1$. Therefore the type distribution $\mu_f^* = \sum_{j=1}^n w^j \mu_f^j = \sum_{j: w_j > 0, \mu_f^j \neq \mathbf{0}} w^j \mu_f^j$ is a convex combination of those μ_f^j's. So we have $\mu_f^* \succsim_f x$ by convexity of $\succsim_f$. This implies $(f, x) \ntrianglerighteq \mu^*$.

Case B: The player i is a worker type θ.

Because $\phi_\theta(f, x) > 0$ implies $x(\theta) > 0$ and $\sum_{j=1}^n w^j \phi_\theta(\mu^j) = 1$ implies $m(\theta) = \sum_{f' \in F} \mu_{f'}^*(\theta)$, to show $(f, x) \ntrianglerighteq \mu^*$, it is sufficient to show $(f', \mu_{f'}^*(\theta)) \succsim_\theta (f, x)$ for each f' with $\mu_{f'}^*(\theta) > 0$.

Take any f' with $\mu_{f'}^*(\theta) > 0$. Note that

$$\mu_{f'}^* = \sum_{j: w^j > 0, \mu_{f'}^j(\theta) > 0} w^j \mu_{f'}^j + \sum_{j: \mu_{f'}^j \neq \mathbf{0}, \mu_{f'}^j(\theta) = 0} w^j \mu_{f'}^j + (1 - \sum_{j: \mu_{f'}^j \neq \mathbf{0}} w^j) \cdot \mathbf{0},$$

i.e., $\mu_{f'}^*$ is a convex combination of $\mu_{f'}^j$'s and $\mathbf{0}$ since $\sum_{j: \mu_f^j \neq \mathbf{0}} w^j \leq 1$. For each j with $w^j > 0$ and $\mu_{f'}^j(\theta) > 0$, we have $\phi_\theta(\mu^j) > 0$ and therefore $\mu^j \trianglerighteq_\theta (f, x)$. By definition of $\trianglerighteq_\theta$, this implies that μ^j is a simple matching. Because $\mu_{f'}^j(\theta) > 0$, the simple matching μ^j is exactly $(f', \mu_{f'}^j)$. Then $(f', \mu_{f'}^j) \trianglerighteq_\theta (f, x)$ implies $(f', \mu_{f'}^j) \succsim_\theta (f, x)$. Because

$$\sum_{j: w^j > 0, \mu_{f'}^j(\theta) > 0} w^j \mu_{f'}^j(\theta) = \mu_{f'}^*(\theta) > 0,$$

there must exist at least one $\hat{j}$ with $w^{\hat{j}} > 0$ and $\mu_{f'}^{\hat{j}}(\theta) > 0$. By the assumption of within-type competition, we have $(f', \mathbf{0}) \succsim_\theta (f', \mu_{f'}^{\hat{j}}) \succsim_\theta (f, x)$ and for each j with $\mu_{f'}^j(\theta) = 0$, we have $(f', \mu_{f'}^j) \succsim_\theta (f', \mu_{f'}^{\hat{j}}) \succsim_\theta (f, x)$. By convexity of $\succsim_\theta$, we have $(f', \mu_{f'}^*) \succsim_\theta (f, x)$. Therefore we have shown that $(f, x) \ntrianglerighteq \mu^*$, which completes the proof of convexity of the matching game. $\square$

Proof of Proposition 4. **First, we show that the matching game is regular.**

(i) $\mathcal{M}$ is compact.

Recall that the set $\mathcal{M}$ of individually rational matchings is

$$\mathcal{M} = \left\{ \mu \in \mathbb{R}_+^{F \times X_+} : \begin{array}{c} \mu(f, x) = 0 \text{ if } (f, x) \text{ is not individually rational,} \\ \sum_{x \in X_+} \mu(f, x) \leq m(f) \text{ for each } f, \\ \text{and } \sum_{(f, x) \in F \times X_+} \mu(f, x) \cdot x(\theta) \leq m(\theta) \text{ for each } \theta \end{array} \right\}.$$

Clearly, $\mathcal{M}$ is closed and bounded, and therefore compact (w.r.t. the standard Euclidean topology).

(ii) $NB(\mu)$ is closed for each nontrivial matching μ.

If μ is nonsimple, the way we define the blocking relation $\sqsupset$ implies $NB(\mu) = \mathcal{M}$, which is a closed set. So we only need to consider the case in which μ is a simple matching. Let the support of μ be $\{(f, x)\}$.

Take a sequence of matchings (μ^k) in $NB(\mu)$ convergent to $\mu^0 \in \mathcal{M}$, and we want to show $\mu^0 \in NB(\mu)$. By definition of s-blocking, $\mu^k \in NB(\mu)$ implies that for each $(f, x') \in F \times X$ with $\mu^k(f, x') > 0$, either (1) $x' \neq \mathbf{0}$ and $x' \succsim_f x$ or (2) for some worker type θ with $x(\theta) > 0$, we have $x'(\theta) = 0$, $\sum_{(f,x)\in F\times X_+} \mu^k(f, x) \cdot x(\theta) = m(\theta)$, and $f'' \succsim_\theta f$ for any (f'', x'') in the support of μ^k with $x''(\theta) > 0$. To show $\mu^0 \in NB(\mu)$, take any (f, x') in the support of μ^0. There exists a subsequence of (μ^k) s.t. $\mu^k(f, x') > 0$ along the subsequence. If condition (1) "$x' \neq \mathbf{0}$ and $x \precsim_f x'$" of $\mu^0 \in NB(\mu)$ does not hold, then condition (2) must hold for every μ^k in the subsequence, which implies that it also holds in the limit μ^0 since Θ is a finite set. Therefore we have $\mu^0 \in NB(\mu)$, which completes the proof of regularity of the matching game.

Second, we show that the matching game is convex.

(i) Let $\mu^* = \sum_{j=1}^n w^j \mu^j$ be a ϕ-convex combination of finitely many matchings $\{\mu^j\}_{j=1}^n$ in $\mathcal{M}$. We need to show that $\mu^* \in \mathcal{M}$.

If (f, x) is not individually rational, $\mu^j \in \mathcal{M}$ implies $\mu^j(f, x) = 0$ and therefore $\mu^*(f, x) = \sum_{j=1}^n w^j \mu^j(f, x) = 0$. Moreover, for each firm type f we have

$$\begin{aligned}
\sum_{x\in X_+} \mu^*(f, x) &= \sum_{x\in X_+} \sum_{j=1}^n w^j \mu^j(f, x) = \sum_{j=1}^n [w^j \sum_{x\in X_+} \mu^j(f, x)] \\
&= m(f) \cdot \sum_{j=1}^n w^j \phi_f(\mu^j) \leq m(f),
\end{aligned}$$

and for each worker type θ we have

$$\begin{aligned}
\sum_{(f,x)\in F\times X_+} \mu^*(f, x) \cdot x(\theta) &= \sum_{j=1}^n [w^j \sum_{(f,x)\in F\times X_+} \mu^j(f, x) \cdot x(\theta)] \\
&= m(\theta) \cdot \sum_{j=1}^n w^j \phi_\theta(\mu^j) \leq m(\theta).
\end{aligned}$$

Therefore, we have $\mu^* \in \mathcal{M}$.

(ii) If $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ s.t. $\phi_i(\mu) > 0$, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, we need to show that $\mu \not\triangleright \mu^*$.

This holds by definition if μ is nonsimple and so we only need to consider the case in which μ is a simple matching. Let the support of μ be $\{(f, x)\}$.

Case A: The player i is the firm type f.

To show $\mu \not\triangleright \mu^*$, by definition of s-block (Definition 7), it is sufficient to show that for each $(f, x') \in F \times X$ with $\mu^*(f, x') > 0$, we have $x' \neq \mathbf{0}$ and $x' \succsim_f x$.

For each $(f, x') \in F \times X$ with $\mu^*(f, x') > 0$, we know that $x' \neq \mathbf{0}$ because $\sum_{j=1}^n w^j \phi_f(\mu^j) = 1$ implies $\sum_{x' \in X_+} \mu^*(f, x') = m(f)$, which in turn implies $\mu^*(f, \mathbf{0}) = 0$. Then we have $\sum_{j=1}^n w^j \mu^j(f, x') = \mu^*(f, x') > 0$ and so there exists j s.t. $w^j > 0$ and $\mu^j(f, x') > 0$. This implies $\phi_f(\mu^j) > 0$ and so $\mu^j \trianglerighteq_f \mu$. By definition of the relation $\trianglerighteq_f$, the matching μ^j must be simple and so its support is exactly $\{(f, x')\}$. Then $\mu^j \trianglerighteq_f \mu$ implies $x' \succsim_f x$.

Case B: The player i is some worker type θ.

Note that $\phi_\theta(\mu) > 0$ implies $x(\theta) > 0$ and that $\sum_{j=1}^n w^j \phi_\theta(\mu^j) = 1$ implies

$$\sum_{(f'', x'') \in F \times X_+} \mu^*(f'', x'') \cdot x''(\theta) = m(\theta).$$

Therefore, to show $\mu \not\triangleright \mu^*$, it is sufficient to show that for each $(f, x') \in F \times X$ with $\mu^*(f, x') > 0$, either (1) $x' \neq \mathbf{0}$ and $x' \succsim_f x$ or (2) $x'(\theta) = 0$ and $f'' \succsim_\theta f$ for each (f'', x'') in the support of μ^* with $x''(\theta) > 0$.

Take any $(f, x') \in F \times X$ with $\mu^*(f, x') > 0$. If $x'(\theta) = 0$, condition (2) above must hold. To see this, take any (f'', x'') in the support of μ^* with $x''(\theta) > 0$. Because $\sum_{j=1}^n w^j \mu^j(f'', x'') = \mu^*(f'', x'') > 0$, there exists some j s.t. $w^j > 0$ and $\mu^j(f'', x'') > 0$. This implies $\phi_\theta(\mu^j) > 0$ and so $\mu^j \trianglerighteq_\theta \mu$. By definition of the relation $\trianglerighteq_\theta$, the matching μ^j must be simple and so its support is exactly $\{(f'', x'')\}$. Then $\mu^j \trianglerighteq_\theta \mu$ implies $f'' \succsim_\theta f$. On the other hand, if $x'(\theta) > 0$, condition (1) must hold. To see this, because $x' \neq \mathbf{0}$,we have $\sum_{j=1}^n w^j \mu^j(f, x') = \mu^*(f, x') > 0$. So there exists j s.t. $w^j > 0$ and $\mu^j(f, x') > 0$. This implies $\phi_\theta(\mu^j) > 0$ and so $\mu^j \trianglerighteq_\theta \mu$. By definition of the relation $\trianglerighteq_\theta$, the matching μ^j must be simple and so its support is exactly $\{(f, x')\}$. Then $\mu^j \trianglerighteq_\theta \mu$ implies $x' \succsim_f x$ since we use the firm's preferences as the tie-breaker by definition of $\trianglerighteq_\theta$. This completes the proof of convexity of the matching game. $\square$

Proof of Proposition 5. **First, we show that the matching game is regular.**

(i) $\mathcal{M}$ is compact.

Recall that the set $\mathcal{M}$ of individually rational matchings is

$$\mathcal{M} = \{\mu \in X^F : \sum_{f \in F} \mu_f \leq m, \mu_f \succsim_f \mathbf{0} \text{ for each } f, \text{ and } \mu_f(\theta) = 0 \text{ if } f \prec_\theta \emptyset\}.$$

By continuity of $\succsim_f$, the set $\mathcal{M}$ is closed. Therefore it is also compact (w.r.t. the standard Euclidean topology) since X^F is bounded.

(ii) $NB(\mu)$ is closed for each nontrivial matching μ.

If μ is nonsimple, the way we define the blocking relation $\sqsupset$ implies $NB(\mu) = \mathcal{M}$, which is a closed set. So we only need to consider the case in which μ is a simple matching (f, x).

Take a sequence of matchings (μ^k) in $NB(f, x)$ convergent to $\mu^0 \in \mathcal{M}$, and we want to show $\mu^0 \in NB(f, x)$. By definition of $\sqsupset$, $\mu^k \in NB(f, x)$ implies that either (a) $\mu^k_f \succsim_f x$ or (b)

$$\mu^k_f(\theta) + m(\theta) - \sum_{f' \in F} \mu^k_{f'}(\theta) + \sum_{f' \in F : f' \prec_\theta f} \mu^k_{f'}(\theta) \leq x(\theta) \tag{5}$$

for some worker type $\theta \in \Theta$ with $x(\theta) > 0$. If there are infinitely many k's that satisfy condition (a) $\mu^k_f \succsim_f x$, by continuity of $\succsim_f$ we have $\mu^0_f \succsim_f x$ in the limit and so $\mu^0 \in NB(\mu)$. If there are only finitely many k's that satisfy condition (a), there must exist infinitely many k's that satisfy condition (b). By finiteness of Θ, there must exists some $\theta \in \Theta$ with $x(\theta) > 0$ s.t. Inequality (5) holds along a subsequence of (μ^k). So the inequality also holds in the limit μ^0 and again we have $\mu^0 \in NB(\mu)$. This completes the proof of regularity of the matching game.

Second, we show that the matching game is convex.

(i) Let $\mu^* = \sum_{j=1}^n w^j \mu^j$ be a ϕ-convex combination of finitely many matchings $\{\mu^j\}_{j=1}^n$ in $\mathcal{M}$. We need to show that $\mu^* \in \mathcal{M}$.

Following exactly the same argument as in the proof of Proposition 3, we can show $\sum_{f \in F} \mu^*_f \leq m$ and$\mu^*_f \succsim_f \mathbf{0}$ for each f. To show that $f \prec_\theta \emptyset$ implies $\mu^*_f(\theta) = 0$, it is sufficient to note that $f \prec_\theta \emptyset$ implies $\mu^j_f(\theta) = 0$ for each j and therefore $\mu^*_f(\theta) = \sum_{j=1}^n w^j \mu^j_f(\theta) = 0$.

(ii) If $\mu^* = \sum_{j=1}^n w^j \mu^j$ is a ϕ-convex combination of $\{\mu^j\}_{j=1}^n$ and $\exists$ player $i \in I$ s.t. $\phi_i(\mu) > 0$, $\sum_{j=1}^n w^j \phi_i(\mu^j) = 1$, and $\mu^j \trianglerighteq_i \mu$ for all j with $w^j > 0$ and $\phi_i(\mu^j) > 0$, we need to show that $\mu \not\sqsupset \mu^*$.

This holds by definition if μ is nonsimple and so we only need to consider the case in which μ is a simple matching. Let μ be (f, x).

Case A: The player i is the firm f.

We can show $(f, x) \not\sqsupseteq \mu^*$ in this case following exactly the same argument as in the proof of Proposition 3.

Case B: The player i is a worker type θ.

Because $\phi_\theta(f, x) > 0$ implies $x(\theta) > 0$ and $\sum_{j=1}^n w^j \phi_\theta(\mu^j) = 1$ implies $m(\theta) = \sum_{f' \in F} \mu^*_{f'}(\theta)$, to show $(f, x) \not\sqsupseteq \mu^*$, it is sufficient to show

$$\mu^*_f(\theta) + \sum_{f' \in F : f' \prec_\theta f} \mu^*_{f'}(\theta) \leq x(\theta). \tag{6}$$

For each j with $w^j > 0$ and $\phi_\theta(\mu^j) > 0$, we have $\mu^j \trianglerighteq_\theta (f, x)$. Therefore μ^j is also a simple matching by definition of $\trianglerighteq_\theta$. Let such μ^j be (f^j, x^j). For each j with $w^j > 0$, $\phi_\theta(\mu^j) > 0$, and $f^j = f$, the relation $(f^j, x^j) \trianglerighteq_\theta (f, x)$ implies $x^j(\theta) \leq x(\theta)$, and so we have

$$\begin{aligned}
\mu^*_f(\theta) = \sum_{j=1}^n w^j \mu^j_f(\theta) = \sum_{j : w_j > 0, \mu^j_f(\theta) > 0, f^j = f} w^j x^j(\theta) \leq \sum_{j : w_j > 0, \mu^j_f(\theta) > 0, f^j = f} w^j x(\theta) \\
\leq x(\theta) \cdot \sum_{j : f^j = f} w^j \leq x(\theta) \cdot \sum_{j=1}^n w^j \phi_f(\mu^j) \leq x(\theta).
\end{aligned}$$

For each j with $w^j > 0$, $\phi_\theta(\mu^j) > 0$, and $f^j \neq f$, the relation $(f^j, x^j) \trianglerighteq_\theta (f, x)$ implies $f^j \succsim_\theta f$, and so we have

$$\begin{aligned}
\sum_{f' \in F : f' \prec_\theta f} \mu^*_{f'}(\theta) = \sum_{j=1}^n [w^j \sum_{f' \in F : f' \prec_\theta f} \mu^j_{f'}(\theta)] = \sum_{j : w_j > 0, \phi_\theta(\mu^j) > 0} [w^j \sum_{f' \in F : f' \prec_\theta f} \mu^j_{f'}(\theta)] \\
= \sum_{j : w_j > 0, \phi_\theta(\mu^j) > 0, f^j \prec_\theta f} w^j x^j(\theta) = 0.
\end{aligned}$$

Combining the two observations above, we have Inequality (6) as desired. This completes the proof of convexity of the matching game. $\square$

Appendix B: Large-firm Labor Market with Compact Set of Worker Types

We can generalize Proposition 5 to allow for a compact set of worker types as in Che, Kim, and Kojima (2019) using the standard argument that a compact set can be approximated arbitrarily well by a finite set.

Let us first adapt the setting in Section 8 to consider a continuum of workers whose type are distributed in a compact metric space Θ according to the Borel measure m. Let X be the set of Borel measure x's with $x \leq m$,[25] and we define a matching $\mu = \{\mu_f\}_{f \in F}$ in the same way as in Section 8. Let $\bar{X} \supset X$ be the set of Borel measure x's on Θ with $x(\Theta) \leq m(\Theta)$. Each firm f has a complete and transitive preference relation $\succsim_f$ on $\bar{X}$ and we assume that $\succsim_f$ is convex and continuous. For each complete and transitive preference relation $\succsim$ on $F \cup \{\emptyset\}$, we let $\Theta_{\succsim}$ be the set of worker type θ's with $\succsim_\theta = \succsim$ and we assume that each $\Theta_{\succsim}$ is a Borel set. Furthermore, we also assume that for each firm f, the set of worker type θ's with $f \succsim_\theta \emptyset$ is closed.[26]

We say that a matching $\mu = \{\mu_f\}_{f \in F}$ is *individually rational* if for each firm f, we have $\mu_f \succsim_f \mathbf{0}$ and $\mu_f(\{\theta \in \Theta : f \prec_\theta \emptyset\}) = 0$. As in Definition 8, a matching μ is *s-blocked* by a simple matching (f, x) if $x \succ_f \mu_f$ and $\mu_f + m - \sum_{f' \in F} \mu_{f'} + \sum_{f' \in F : f' \prec_\theta f} \mu_{f'} \geq x$, and a matching is *stable* if it is individually rational and not s-blocked by any simple matching.

Now we show the existence of stable matchings. For each positive integer n, find a finite set $\Theta^n_{\succsim}$ in $\Theta_{\succsim}$ s.t. the distance from each $\theta \in \Theta_{\succsim}$ to $\Theta^n_{\succsim}$ is less than $1/n$.[27] Partition $\Theta_{\succsim}$ as $\bigcup_{\theta \in \Theta^n_{\succsim}} S^n(\theta)$ s.t. each $S^n(\theta)$ is a Borel set that lies in the open ball $B_{1/n}(\theta) \cap \Theta_{\succsim}$. Let $\Theta^n := \bigcup_{\succsim} \Theta^n_{\succsim}$ and then $\{S^n(\theta)\}_{\theta \in \Theta^n}$ is a finite partition of the worker type space Θ.

Define the Borel measure m^n on Θ with the finite support Θ^n as

$$m^n(\Theta') := \sum_{\theta \in \Theta' \cap \Theta^n} m(S^n(\theta))$$

for each Borel set Θ'. By construction, we have $m^n(\Theta_{\succsim}) = m(\Theta_{\succsim})$ for each $\succsim$ and so $m^n \in \bar{X}$. Furthermore, the sequence of measures (m^n) converges to m. To see this, for

[25] When comparing two Borel measures x and x' on Θ, the inequality $x \leq x'$ means that $x(\Theta') \leq x'(\Theta')$ for each Borel set Θ'.

[26] This assumption will be satisfied if, for example, the workers' preferences are represented by a system of utility functions $u(f, \theta)$ continuous in θ

[27] This can be done because the collection of open balls $B_{1/n}(\theta)$ with $\theta \in \Theta^n_{\succsim}$ is an open cover of the closure of $\Theta^n_{\succsim}$, which is compact.

each continuous function f on Θ, we have

$$|\int_\Theta f dm^n - \int_\Theta f dm| = |\sum_{\theta\in\Theta^n} f(\theta)m(S^n(\theta)) - \sum_{\theta\in\Theta^n}\int_{S^n(\theta)} f(\theta')dm|$$
$$\leq \sum_{\theta\in\Theta^n}\int_{S^n(\theta)} |f(\theta') - f(\theta)|dm,$$

which converges to 0 as $n \to \infty$ because $S^n(\theta) \subset B_{1/n}(\theta)$ by construction and the continuous function f on the compact set Θ is uniformly continuous.

For each m^n, we invoke Proposition 5 to find a stable matching μ^n under the discrete worker type distribution m^n. Because each μ_f^n lies in the compact metrizable space $\bar{X}$, the sequence of matchings (μ^n) contains a subsequence of (μ^{n_k}) convergent to some μ^*. We show that the limit μ^* is a stable matching under the worker type distribution m. To see this, first note that μ^* is indeed a feasible matching, i.e., $\sum_f \mu_f^* \leq m$, because $\sum_f \mu_f^{n_k} \leq m^{n_k}$ holds for each k. Second, μ is individually rational because $\mu_f^{n_k} \succsim_f \mathbf{0}$ for each k implies $\mu_f^* \succsim_f \mathbf{0}$ since $\succsim_f$ is continuous, and $\mu_f^{n_k}(\{\theta \in \Theta : f \prec_\theta \emptyset\}) = 0$ for each k implies $\mu_f^*(\{\theta \in \Theta : f \prec_\theta \emptyset\}) = 0$ by Portmanteau theorem since the set $\{\theta \in \Theta : f \prec_\theta \emptyset\}$ is open.

Now it is only left to show that μ^* is unblocked. Suppose to the contrary that μ^* is s-blocked by some simple matching (f, x). Define the measure x^n on Θ as $x^n(\Theta') := \sum_{\theta\in\Theta'\cap\Theta^n} x(S^n(\theta))$ for each Borel set Θ'. Also, let $z^n := \mu_f^n + m^n - \sum_{f'\in F}\mu_{f'}^n + \sum_{f'\in F: f'\prec_\theta f}\mu_{f'}^n$ be the type distribution of workers available to firm f to form a block under μ^n. Similarly, in the limit we let $z^* := \mu_f^* + m^* - \sum_{f'\in F}\mu_{f'}^* + \sum_{f'\in F: f'\prec_\theta f}\mu_{f'}^*$. Let $z^n \wedge x^n$ denote the *meet* of z^n and x^n, i.e., the greatest lower bound of z^n and x^n.[28] Because $z^{n_k} \to z^*$, $x^{n_k} \to x$, and $z^* \geq x$ since (f, x) s-blocks μ^*, we have $(z^{n_k} \wedge x^{n_k}) \to x$. Again since (f, x) s-blocks μ^*, we have $x \succ_f \mu_f^*$, which implies $(z^{n_k} \wedge x^{n_k}) \succ_f \mu_f^{n_k}$ for sufficiently large k because $\succsim_f$ is continuous. Then the simple matching $(f, z^{n_k} \wedge x^{n_k})$ s-blocks $\mu_f^{n_k}$, which contradicts to the construction of μ^{n_k} as a stable matching under the worker type distribution m^n.

Therefore, the limiting matching μ^* is stable under the worker type distribution m.

[28] Formally, the meet $z^n \wedge x^n$ is defined as

$$(z^n \wedge x^n)(\Theta') := \inf_{\text{Borel set } E \text{ in } \Theta} z^n(\Theta' \cap E) + x^n(\Theta' \backslash E)$$

for each Borel set Θ'.

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