Highlights

A characterization of preference domains that are single-crossing and maximal Condorcet

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- It is shown that (up to relabelling of the alternatives) there are exactly two single-crossing maximal Condorcet domains.
- Single-crossing maximal Condorcet domains are shown to have a relay representation and characterized in terms of inversion triples.

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A characterization of preference domains that are single-crossing and maximal Condorcet

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Abstract

We show that a preference domain is single-crossing and maximal Condorcet if and only if it can be represented as a *relay*, a structure that is simple to construct and verify. Using this characterization, we find that there are at most two domains that are single-crossing and maximal Condorcet, and we also find another characterization of such domains in terms of inversion triples.

Keywords: Condorcet domain, single-crossing domain
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1. Introduction

A domain is said to be *single-crossing* if the preferences in it can be linearly ordered so that along this ordering the relative positions of any pair of alternatives swap at most once. This property is economically intuitive and easily checked in applications, e.g., voting models of redistributive income taxation and trade union bargaining behavior [1, 2], whereas [3] provided an efficient way to verify this property for domains in general.

Single-crossing domains have many nice properties that have attracted the attention of researchers. In particular, single-crossing domains have the

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Condorcet property, i.e., pairwise majority voting never admits cycles if preferences of a group of agents are from such a domain. In fact, single-crossing further implies that there is always an agent in the group whose preference coincides with the group preference aggregated by means of the pairwise majority voting — this fact is known as the Representative Voter Theorem [4, 5, 6]. Moreover, the collective choice predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism [7], among the many social choice rules implementable in dominant strategies on single-crossing domains [8].

Recent research has revealed that understanding single-crossing domains could be crucial to understanding Condorcet domains in general. Indeed, Galambos and Reiner [9] proved that any connected maximal Condorcet domain of maximal width is a union of single-crossing domains. An important question of the same spirit, then, is the following: when is a maximal Condorcet domain also by itself single-crossing? One answer to this question was given by Puppe and Slinko [10] who gave a characterization in terms of a property called the *pairwise concatenation condition*.

In this paper, we provide an alternative combinatorial characterization which allows a deeper understanding of single-crossing and maximal Condorcet domains.[4] Specifically, it allows us to conclude that a maximal Condorcet domain is single-crossing if and only if it can be represented by a combinatorial structure that we call *relay*, which is rather intuitive to understand and easy to construct and verify. With the help of our characterization we show that, when there are more than two alternatives, there are essentially only two domains that are single-crossing and maximal Condorcet. In addition, the relay representation makes it easy to construct the set of inversion triples that characterizes a single-crossing and maximal Condorcet domain.

2. Preliminaries

Consider the standard social choice environment where there is a set $X = \{1, 2, \ldots, n\}$ of n alternatives. We will focus on strict preferences, i.e., linear orders on X, and we denote $\mathcal{L}(X)$ as the set of all strict preferences on X. Sometimes we represent a preference by its implied ranking of alternatives,

[4] A combinatorial characterization of single-crossing domains in terms of forbidden minors was given in [3].

i.e., the ranking $ijk\dots$ represents the preference $\succ$ where $i \succ j \succ k \succ \dots$. A *domain* is a subset of $\mathcal{L}(X)$.

A domain $\mathcal{D}$ is said to be a *single-crossing domain* if there is an ordering $(\succ_1, \dots, \succ_{|\mathcal{D}|})$ of preferences in $\mathcal{D}$ such that $i \succ_1 j$ implies either $i \succ_s j$ for every s, or there is a unique k where $i \succ_s j$ for every $s \leq k$ and $j \succ_s i$ for every $s > k$. Simply put, traveling along $\succ_1, \succ_2, \dots$ the relative positions of i and j swap at most once. In addition, if $\mathcal{D}$ is not a proper subset of another single-crossing domain, then we say it is a *maximal single-crossing domain*.

A domain $\mathcal{D}$ is said to be a *Condorcet domain* if Condorcet voting cycles do not arise given $\mathcal{D}$, i.e., for any group of agents whose preferences are from $\mathcal{D}$ and any three alternatives i, j, k, if more than half of the agents prefer i to j and more than half of the agents prefer j to k, then more than half of the agents prefer i to k. In addition, if $\mathcal{D}$ is not a proper subset of another Condorcet domain, then we say it is a *maximal Condorcet domain*.

Let us introduce a few more concepts that will help illustrate the structure of $\mathcal{L}(X)$ and its relation to single-crossing and Condorcet domains. For any preference $\succ \in \mathcal{L}(X)$, we say that a pair of alternatives (i, j) is an *inversion* in this preference if $i < j$ but $j \succ i$. Thus there is no inversion in the preference $12\dots n$ whereas every pair of alternatives is an inversion in $n(n-1)\dots 1$. For two preferences $\succ, \succ' \in \mathcal{L}(X)$, we say that $\succ$ *covers* $\succ'$ if the set of inversions in $\succ'$ is a subset of the set of inversions in $\succ$. The covering relation is obviously reflexive, antisymmetric and transitive, and it hence induces a partial order on $\mathcal{L}(X)$, which is known as the *weak Bruhat order*. With the weak Bruhat order, $\mathcal{L}(X)$ becomes a lattice [11] (Theorem 14.4).

It is easy to verify that any maximal chain in $\mathcal{L}(X)$ (on the lattice given by the weak Bruhat order) is a maximal single-crossing domain. Also, every maximal single-crossing domain is a maximal chain in $\mathcal{L}(X)$ up to relabelling of the alternatives. Indeed, given any maximal single-crossing domain $(\succ_1, ..., \succ_m)$ (ordered so that the single-swap condition holds), we can rename the alternatives so that the first preference is $\succ'_1 = 12\dots n$ and the last is $\succ'_m = n(n-1)\dots 1$, then what we obtain is a maximal chain $(\succ'_1, \dots, \succ'_m)$ in $\mathcal{L}(X)$ where $m = \frac{1}{2}n(n-1) + 1$ and each $\succ'_s$ is covered by the next $\succ'_{s+1}$. Clearly, this chain is characterized by a sequence of $m-1$ pairs of alternatives

$$(i_1, j_1),\ (i_2, j_2),\ \dots, (i_{m-1}, j_{m-1}) \tag{1}$$

from the set $\{(i, j) \mid 1 \leq i < j \leq n\}$. The pair (i_s, j_s) means that i_s and j_s are neighbors in $\succ_s$ and $\succ_{s+1}$, $i_s \succ_t j_s$ for $t = 1, \dots, s$, and $j_s \succ_t i_s$ for

$t = s+1, \ldots, n$, while all other relations between alternatives coincide in $\succ_s$ and $\succ_{s+1}$. Roughly speaking, the passage from $\succ_s$ to $\succ_{s+1}$ is a swap of neighbors i_s and j_s.

The following theorem is implied from [9] Theorem 2 and is explicitly stated in [10] Theorem 9.

Theorem 1. *A maximal single-crossing domain $\mathcal{D}$ is a maximal Condorcet domain if and only if the sequence* (1) *characterizing $\mathcal{D}$ satisfies the following:*

$$\{i_s, j_s\} \cap \{i_{s+1}, j_{s+1}\} \neq \emptyset \text{ for every } s \in \{1, 2, \ldots, m-1\}. \tag{2}$$

The condition (2) is called the *pairwise concatenation condition* in [10]. Theorem 1 is the first known characterization of domains that are single-crossing and maximal Condorcet, and it will also be an important building block for our characterization.

Let us introduce another concept that our result will shed some light on. Along a maximal chain on $\mathcal{L}(X)$ the preference $12\ldots n$ is transformed into $n\ldots 21$ by a sequence of swaps of neighboring alternatives. Focusing on any three alternatives $i < j < k$, there are two ways their relative rankings can be transformed from ijk ito kji along the chain, namely

$$ijk \to jik \to jki \to kji \quad \text{or} \quad ijk \to ikj \to kij \to kji.$$

If it is the second transformation that takes place, then $[i, j, k]$ is called an *inversion triple* of the maximal chain. It is known that two maximal chains can have the same set of inversion triples, in which case the two chains are said to be *equivalent*. By [9] Theorem 2, a domain that contains a maximal chain C is a maximal Condorcet domain if and only if it is the union of all maximal chains equivalent to C. Therefore, inversion triples provide a very succinct and convenient characterization of maximal Condorcet domains.

As an example, let us consider the maximal chain whose preferences are represented as columns of the following matrix

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It can be characterized by the sequence of swapping pairs

$$(1,2),\ (1,3),\ (1,4),\ (3,4),\ (2,4),\ (2,3).$$

Since the pairwise concatenation condition is satisfied, this single-crossing domain is a maximal Condorcet domain. It is not too difficult to see that it is characterized by a single inversion triple $[2, 3, 4]$. Later on, we will show that our main result provides an easy way of finding the characterizing set of inversion triples of a domain that is single-crossing and maximal Condorcet.

3. Relays

We will show that a domain is single-crossing and maximal Condorcet if and only if it can be represented by a structure that we call a *relay*. Let us first use an example to illustrate what a relay looks like. In this example $X = \{1, 2, \ldots, 7\}$. The domain is represented by the following matrix where each column corresponds to a preference.

$$\begin{bmatrix}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
2 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 7 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & 6 & 6 & 6 \\
3 & 3 & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 7 & 3 & 3 & 2 & 4 & 4 & 4 & 4 & 6 & 3 & 4 & 4 & 5 \\
4 & 4 & 4 & 1 & 5 & 5 & 5 & 5 & 7 & 4 & 4 & 4 & 4 & 2 & 5 & 5 & 6 & 4 & 4 & 3 & 5 & 4 \\
5 & 5 & 5 & 5 & 1 & 6 & 6 & 7 & 5 & 5 & 5 & 5 & 5 & 5 & 2 & 6 & 5 & 5 & 5 & 5 & 3 & 3 \\
6 & 6 & 6 & 6 & 6 & 1 & 7 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
7 & 7 & 7 & 7 & 7 & 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

The domain is a maximal chain satisfying the pairwise concatenation condition, and is hence single-crossing and maximal Condorcet. In addition, with the help of the red-coloring, it is not difficult to see that the left-to-right procession of preferences follows a distinct pattern that leaves behind an undulating trajectory like a damped wave. In particular, focusing on the red-colored alternatives, we see that the procession starts with the movement of 1 that keeps going down from the top until it reaches the bottom. Then 7, which occupies the bottom just before, as if having received a relay baton from 1 as they meet, starts moving up until it reaches the top. As 7 reaches the top, the then top alternative, 2, starts to move down. However, instead of stopping at the bottom, 2 stops at second-to-bottom, handing the baton to the then second-to-bottom alternative, 6, which starts to go up until reaching second-to-top. This to-and-fro relay run continues, each leg ending with the initial kth-to-top alternative reaching the kth-to-bottom position, or the kth-to-bottom alternative reaching the kth-to-top position, until, eventually, the initial ranking is reversed. The red trajectory is undulating because of the to-and-fro relay motion, and it is damped because a later runner covers a shorter distance than an earlier runner.

We call such a procession of preferences a *top-down relay*, because it starts with the top alternative going down. In general, a top-down relay with n alternatives is a sequence of preferences $(\succ_1, ..., \succ_m)$ such that the procession of $\succ_i$ as i grows follows a pattern analogous to the example: First, the initial top alternative moves down, in each step swapping with the alternative below, until reaching the bottom; then, the initial bottom alternative moves up, in each step swapping with the alternative above, until reaching the top; then, the initial second-to-top alternative moves down until reaching the second-to-bottom; then, the initial second-to-bottom alternative moves up until reaching second-to-top, and so on so forth, until the initial preference is reversed.

A more formal definition would be the following: A sequence of preferences $(\succ_1, ..., \succ_m)$ over n alternatives is a *top-down relay* if and only if it can be represented (up to relabelling the alternatives) as the matrix $R_n(1, ..., n)$ recursively defined as follows:

$$R_1(1) = [1], \qquad R_2(1,2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\left[\begin{array}{ccccccc|ccc|}
\color{red}{1} & 2 & \cdots & 2 & 2 & \cdots & 2 & \color{red}{n} & \cdots & n \\ \cline{8-10}
2 & \color{red}{1} & \cdots & 3 & 3 & \cdots & \color{red}{n} & & & \\
3 & 3 & \cdots & 4 & 4 & \cdots & 3 & & & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & R_{n-2}(2,\ldots,n-1) & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & & \\
n-1 & n-1 & \cdots & \color{red}{1} & \color{red}{n} & \cdots & n-1 & & & \\ \cline{8-10}
n & n & \cdots & \color{red}{n} & \color{red}{1} & \cdots & 1 & 1 & \cdots & 1
\end{array}\right]. \tag{3}$$

It is straightforward to construct an analogous procession of preferences that starts with the bottom alternative moving up, then followed by the top alternative moving down, and so on so forth. We call such a procession of preferences a *bottom-up relay*. There is an obvious symmetry between a top-down and a bottom-up relay: one can be obtained by reversing every preference (column) of the other. Top-down relays and bottom-up ones are collectively called *relays*. We say that domain $\mathcal{D}$ *has a relay representation* if there is an ordering $(\succ_1, \ldots, \succ_m)$ of preferences in it that is a relay.

4. Results

We are ready to report here our results, whereas the proofs are found in the Appendix. First, the main result:

Theorem 2. *A domain $\mathcal{D}$ is single-crossing and maximal Condorcet if and only if it has a relay representation.*

The main result allows us to obtain two further observations about single-crossing domains. The first observation is that, when there are at least three alternatives, there are only two domains that are single-crossing and maximal Condorcet; if there are fewer alternatives, then there is only one such domain. This is because, up to relabelling of the alternatives, there is a unique top-down relay and a unique bottom-up relay. Moreover, in the case there are fewer than three alternatives the two relays are the same.

Corollary 1. *If $n \leq 2$, then $\mathcal{L}(X)$ is the unique single-crossing and maximal Condorcet domain. If $n \geq 3$, then there are, up to relabeling the alternatives, exactly two single-crossing and maximal Condorcet domains, one represented by a top-down relay and the other by a bottom-up relay.*

The second observation gives a characterization of single-crossing and maximal Condorcet domains in term of inversion triples.

Corollary 2. *A domain $\mathcal{D}$ over n alternatives is single-crossing and maximal Condorcet if and only if, up to relabelling the alternatives, its set of inversion triples is $T_n^{\downarrow}(1,2,...,n)$ or $T_n^{\uparrow}(1,2,...,n)$ recursively defined as follows:*

$$\begin{aligned}
&T_1^{\downarrow}(1) = T_1^{\uparrow}(1) = T_2^{\downarrow}(1,2) = T_2^{\uparrow}(1,2) = \emptyset,\\
&T_n^{\downarrow}(1,2,\ldots,n) = T_{n-2}^{\downarrow}(2,3,\ldots,n-1) \cup \{[i,i,n] \mid i < j < n\},\\
&T_n^{\uparrow}(1,2,\ldots,n) = \{[i,j,k] \mid 1 \leq i < j < k \leq n\} \backslash T_n^{\downarrow}(1,2,\ldots,n).
\end{aligned}$$

Specifically, if its set of inversion triples is $T_n^{\downarrow}(1,2,...,n)$, then $\mathcal{D}$ has a top-down relay representation; else it has a bottom-up relay representation.

Corollary 2 directly follows the recursive definition of relays. Given this corollary, it is easy to see that a top-down relay with n alternatives has

$$\sum_{i\geq 1, n-2i\geq 2} \binom{n-2i}{2}$$

inversion triples and a bottom-up relay with n alternatives has

$$\binom{n}{3} - \sum_{i\geq 1, n-2i\geq 2} \binom{n-2i}{2} = \sum_{i\geq 1, n-2i+1\geq 2} \binom{n-2i+1}{2}$$

inversion triples.

5. Conclusion

We characterize domains that are single-crossing and maximal Condorcet in terms of relays, which implies there are (up to relabelling of the alternatives) exactly two such domains and which also helps us find another characterization of such domains in terms of inversion triples. Our result is equivalent to a characterization of Condorcet domains whose associated median graph (as defined in [10]) is a line graph. It would be interesting to enumerate and characterize Condorcet domains whose median graph has a more complicated structure.

Appendix: proof of Theorem 2

If $\mathcal{D}$ has a relay representation, then it is easy to verify that $\mathcal{D}$ is a maximal single-crossing domain that satisfies the pairwise concatenation condition. Thus, by Theorem 1, $\mathcal{D}$ is also a maximal Condorcet domain.

Now we prove the other direction. Suppose $\mathcal{D}$ is a single-crossing and maximal Condorcet domain over alternatives $X = \{1, 2, ..., n\}$. It follows that $\mathcal{D}$ contains exactly one pair of reversed preferences. Without loss of generality (by relabelling of the alternatives) assume they are $12\ldots n$ and $n(n-1)\ldots 1$. Let P be the matrix where preferences in $\mathcal{D}$ are written as columns in such an order that the neighboring swapping pairs are linked as in (2). We will show that P is a relay. A couple of observations are helpful.

Lemma 1. *P has one of the following submatrices:*

$$Q_1 = \begin{bmatrix} 1 & \star & \cdots & \star & \star & \cdots & \star & n \\ \star & 1 & \cdots & \star & \star & \cdots & n & \star \\ \star & \star & \cdots & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & 1 & n & \cdots & \star & \star \\ n & n & \cdots & n & 1 & \cdots & 1 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 & n & \cdots & n & n \\ \star & \star & \cdots & n & 1 & \cdots & \star & \star \\ \star & \star & \cdots & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & n & \cdots & \star & \star & \cdots & 1 & \star \\ n & \star & \cdots & \star & \star & \cdots & \star & 1 \end{bmatrix}$$

where all the columns are the same after the removal of 1 *and* n.

Proof. Consider the corresponding sequence of swapping pairs (1) which we know has all possible pairs of distinct alternatives and any two neighboring pairs are linked, i.e., satisfy (2). Let us consider the last column of P such that 1 occupies the top. That column is thus

$$[1, a_1, \ldots, a_{n-2}, a_{n-1}]^T. \tag{4}$$

In the next step (proceeding to the column next to the right), 1 will swap with a_1. Claim that in the subsequent steps, 1 goes straight to the bottom (each step moving down one position). Indeed, suppose at some point we had the column

$$[a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}]^T.$$

where 1 has just swapped with a_{i-1}. If 1 does not swap with a_i in the next step, then it will be a_{i-1} swapping with a_{i-2}. If so, then 1 will then never swap with a_i since for that to happen 1 must be involved in the previous swap with one of $a_1, \ldots, a_{i-1}$, but this is impossible since 1 has already swapped with all these alternatives. Since 1 has to reach the bottom eventually, it follows that the next step must be 1 swapping with a_i, and by induction 1 has to go down continuously to the bottom.

In a similar argument, we can now show that n has to continuously go all the way up to the top once it starts moving. Thus if 1 starts to move before n, then P has Q_1 as a submatrix, whereas if n starts to move before 1 then P has Q_2 as a submatrix. □

Lemma 2. *P has a submatrix $Q \in \{Q_1, Q_2\}$ that occupies either the leftmost $2n-1$ columns or the rightmost $2n-1$ columns.*

Proof. Suppose Q_1 is a submatrix of P. (The case where Q_2 is a submatrix of P can be established in a similar argument.) If the first column of Q_1 (given in (4)) is not the first column in P, then the previous column was

$$[1, a_2, a_1, \ldots, a_{n-2}, \underbrace{a_{n-1}}_{=n}]^T,$$

i.e., the previous swap was between a_1 and a_2. Similarly, if the rightmost column of Q_1

$$[n, a_1, a_2, \ldots, a_{n-2}, 1]^T.$$

is not the last column in P, then the next swap must also be between a_1 and a_2. However, a_1 and a_2 cannot swap more than once, hence either the first column of Q_1 is the first column in P or the last column of Q_1 is the last column in P. □

Now we are ready to show that P is a relay. Suppose Q_1 occupies the leftmost $2n-1$ columns of P. We want to show that P is a top-down relay, i.e. it satisfies the recursive definition given in (3). Consider the inductive hypothesis that for any $k < n$, a matrix with k rows is a top-down relay if it satisfies the following: (a) it corresponds to a maximal chain on $\mathcal{L}(\{i, i+1, ..., i+k-1\})$ for some $i \in \mathbb{N}$, (b) the columns are in such an order that the neighboring swapping pairs are linked as in (2), and (c) in the leftmost $2k-1$ columns the top alternative goes all the way down then followed the bottom alternative goes all the way up as analogous to Q_1. This hypothesis is obviously true for $k = 1$ or $k = 2$.

Since Q_1 occupies the leftmost part of P, the first column of Q_1 is $[1, 2, \ldots, n]^T$. All the remaining columns to the right of Q_1 will have n on the top and 1 at the bottom, and clearly, right after Q_1 the first pair to swap is $[2, 3]$, i.e. 2 starts to move down. Consider the submatrix P' of P enframed by n's on the top and 1's at the bottom: it obviously corresponds to a maximal chain on $\mathcal{L}(\{2, 3, ..., n-1\})$ and the columns are ordered so that the neighboring swapping pairs are linked. Moreover, since 2, the top alternative among $\{2, 3, ..., n-1\}$, is the first to move in P', by an argument analogous to that used in Lemma 1, the leftmost $2n-3$ columns of the P' is analogous to Q_1 (2 moves all the way down, then $n-1$ moves all the way up). Therefore, by the inductive hypothesis P' is a top-down relay, i.e. $P' = R_{n-2}(2, \ldots, n-1)$ (defined in (3)). This implies that $P = R_n(1, 2, ..., n)$. Thus P is a top-down relay. If, on the other hand, Q_1 occupies the rightmost $2n-1$ columns of P, then it is clear that after relabelling every alternative $i = 1, ..., n$ as $n+1-i$ and listing the preferences in a reverse order, the resulting matrix is also $R_n(1, 2, ..., n)$, a top-down relay. If Q_2 is a submatrix of P, then P is shown to be a bottom-up relay in an analogous argument.

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